Lower Runtime Bounds for Integer Programs

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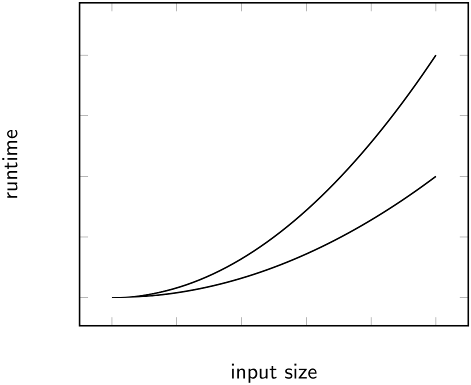
June 27, 2016

Lower Bounds?



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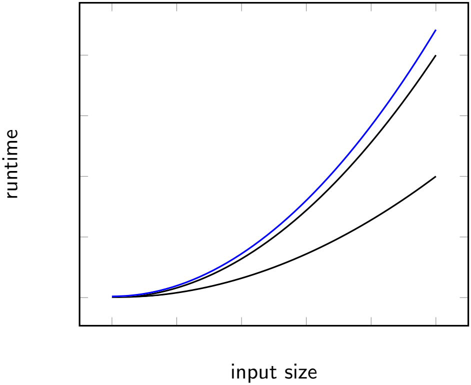

- worst case upper bounds

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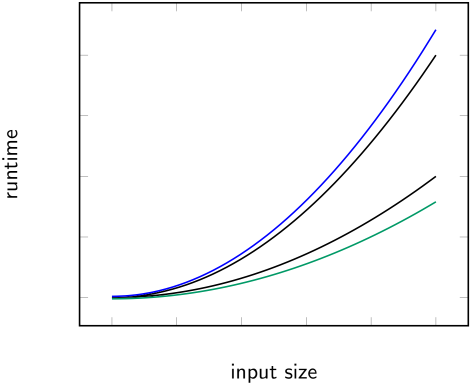

- worst case upper bounds
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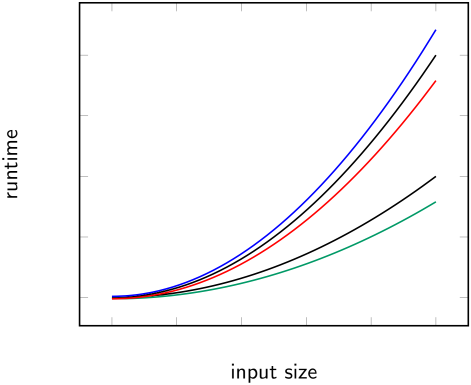

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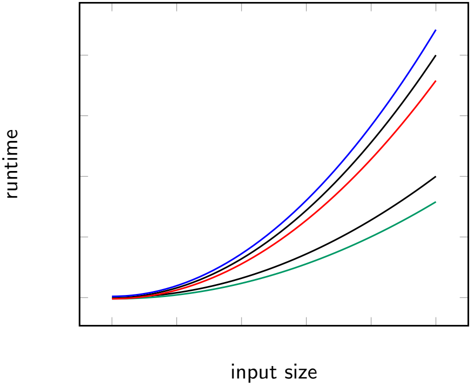

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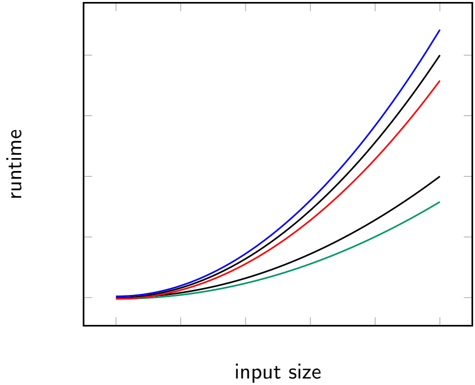
Why?

- *tight* bounds

Lower Bounds?



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Why?

- *tight* bounds
- identify attacks

What kind of programs?

non-recursive integer programs:

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z = y
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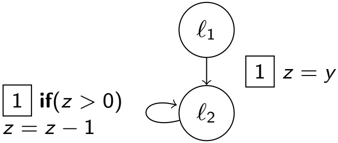

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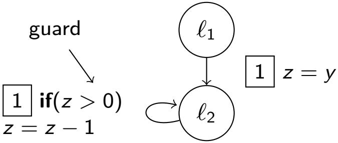

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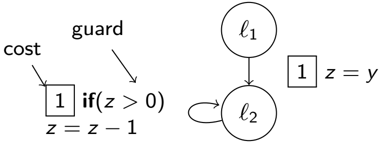

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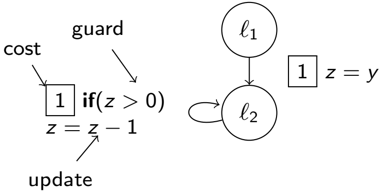

The Technique

- step 1: underapproximating program simplification

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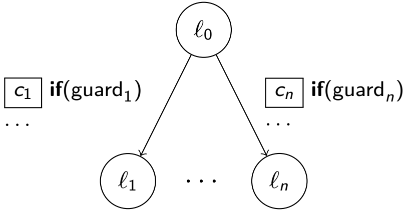

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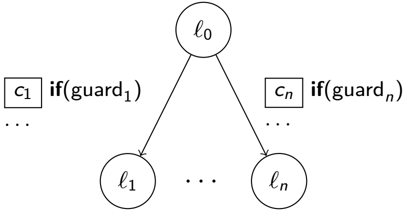

- step 2: infer asymptotic lower bound

Program Simplification

Acceleration and *Chaining*

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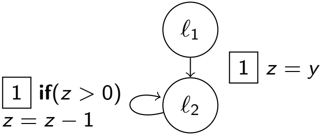
Acceleration and *Chaining*

- *accelerate* simple loops



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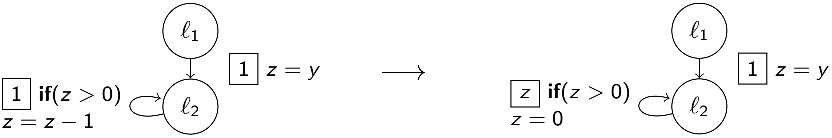

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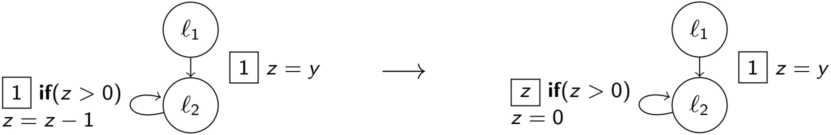
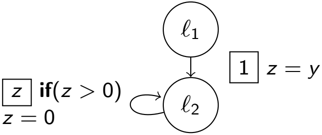


- *chain* subsequent transitions



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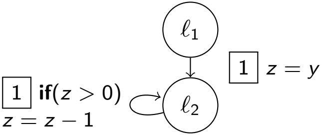

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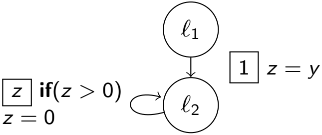

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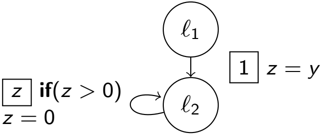

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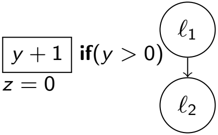

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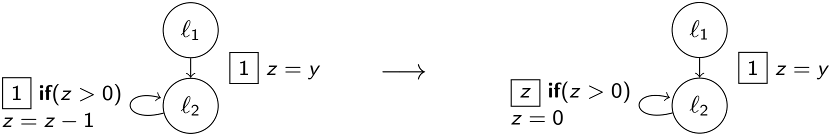

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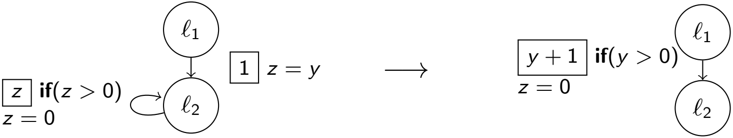
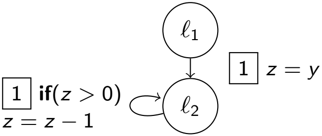


- iterate

Loop Acceleration: Challenges



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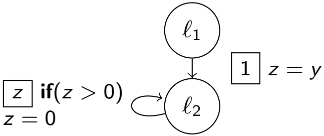

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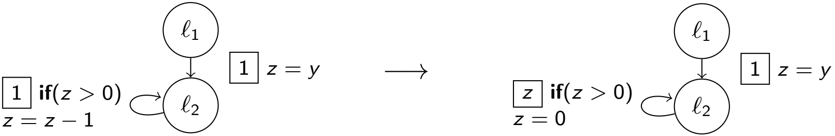

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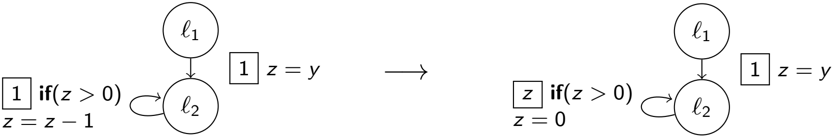

- What's the result?

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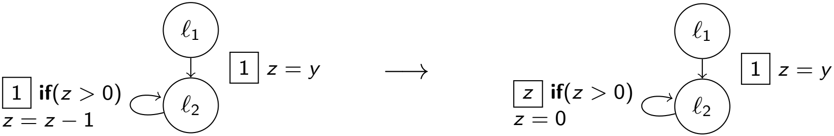

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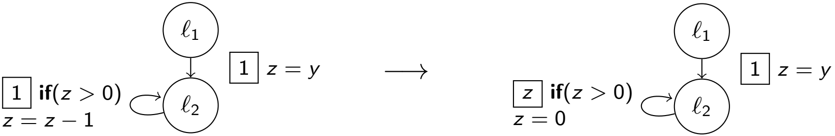

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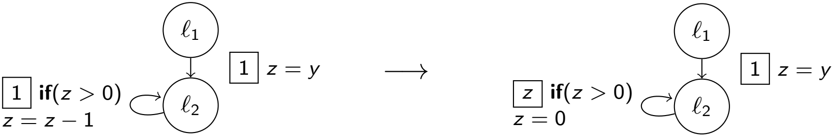

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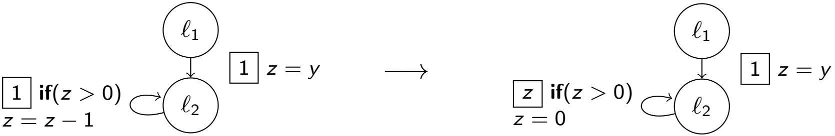

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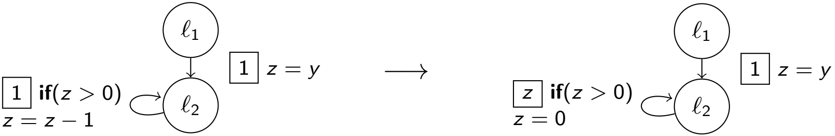

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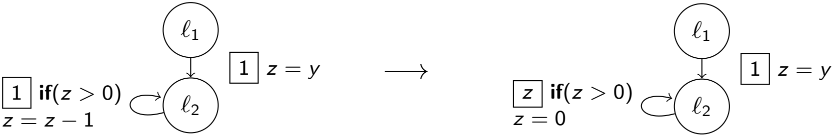

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- How many repetitions?
  - use *metering functions*

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$\Rightarrow$ transition can be applied at least b times

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$$\text{guard} \Rightarrow b > 0 \text{ and guard} \Rightarrow \text{update}(b) \leq b - 1$$

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Metering Functions

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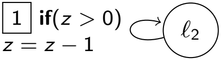
Example



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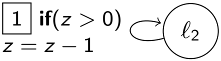

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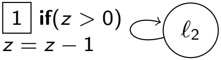

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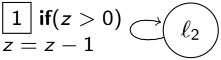

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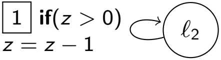

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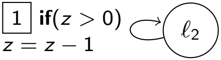

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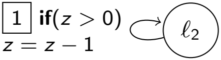

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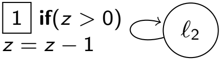

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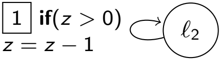

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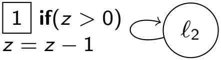

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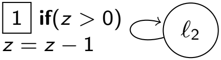

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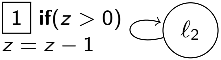

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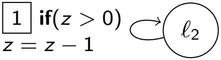

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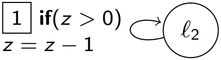

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- $z \leq 0 \Rightarrow z \leq 0$ ✓
  $z > 0 \Rightarrow z - 1 \geq z - 1$ ✓

- $z, z+1, \ldots$
- $2 \cdot z, 3 \cdot z + 1, \ldots$
- $z \leq 0 \Rightarrow z + 1 \leq 0$ ↯
  $z > 0 \Rightarrow 2 \cdot (z-1) \geq 2 \cdot z - 1$ ↯

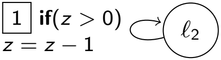
finding them:

Metering Functions

b is a *metering function* iff

$$\neg \text{guard} \Rightarrow b \leq 0 \text{ and guard} \Rightarrow \text{update}(b) \geq b - 1$$

Example



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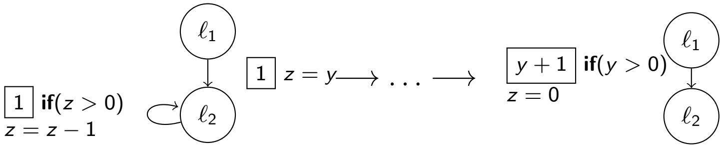
finding them: just like ranking functions

Program Simplification

Algorithm

- while there is a path of length > 1



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- while there is a path of length > 1
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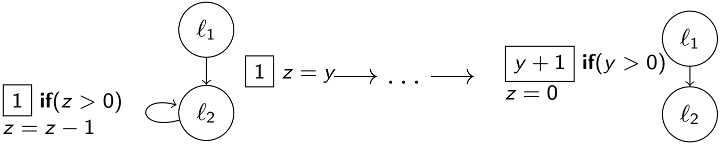
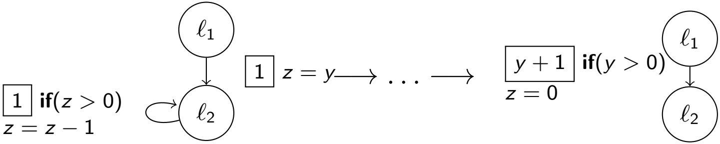


Program Simplification

Algorithm

- while there is a path of length > 1
  - accelerate simple loops
  - chain subsequent transitions

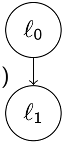


Simplified Programs

$\boxed{x^2 - y}$

$\textbf{if}(0 < x < 10 \land y + x < 0)$

$\dots$



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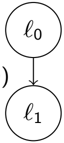

- inferring lower bound still non-trivial

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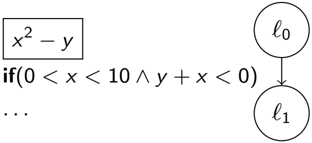

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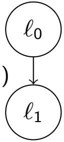

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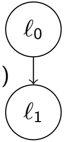

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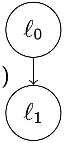

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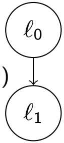

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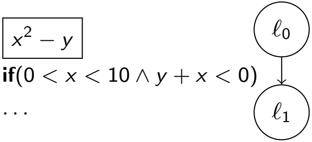

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Example

- $\boldsymbol{v}_n = \{x/1, y/-n\}$ satisfies guard for $n \geq 2$
- $\boldsymbol{v}_n(x^2 - y) = 1 + n \implies \Omega(n)$

Example

goal: infer $\boldsymbol{v}_n = \{x/1, y/-n\}$

$\boxed{x^2 - y}$

if$(x > 0 \land 10 - x > 0 \land -y - x > 0)$

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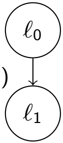

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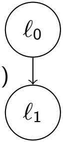

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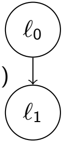

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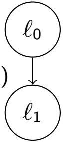

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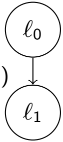

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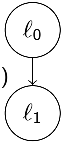

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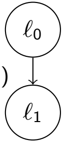

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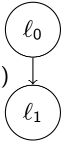

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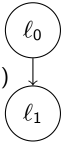

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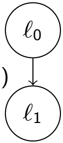

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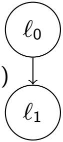

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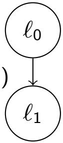

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$\mathcal{O}(1)$	(132)	–	–	–	–	–	–
$\mathcal{O}(n)$	45	125	–	–	–	–	–
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$\mathcal{O}(n^3)$	2	–	–	3	–	–	–
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- non-trivial bounds: 78%, tight bounds: 67%

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