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Exercises *Functional Programming* – Sheet 7

Solutions will be collected until Thursday, Dec 5, 2002 in the exercise course.

Exercise 1 (2 points)

Let $\sqsubseteq_{D_1}$ and $\sqsubseteq_{D_2}$ be complete partial orderings on D_1 and D_2 respectively, and let $f : D_1 \to D_2$ be a monotonic function such that for each chain S of D_1 we have $f(\sqcup S) \sqsubseteq \sqcup f(S)$. Show that f is continuous.

Exercise 2 (2 + 1 + 2 points)

We consider the domain $D = 2^{\mathbb{Z}}$ and the partial ordering $\subseteq$ on D.

(a) What is the lub of a chain $\{M_1, M_2, \ldots\}$? Show that $\subseteq$ is a cpo on D.

(b) Give an example of an infinite chain in $(D, \subseteq)$.

(c) Give an example of a monotonic, non-continuous function $f : D \to D$. Prove that your function is monotonic and not continuous. (Perhaps you can use the chain in part (b) to show that your function is not continuous)

Exercise 3 (1 point)

Let D_1, D_2, and D_3 be domains with cpo's $\sqsubseteq_1, \sqsubseteq_2, \sqsubseteq_3$, respectively. Let $f : D_1 \to D_2$ and $g : D_2 \to D_3$ be continuous functions. Show that the composition $g \circ f : D_1 \to D_3$ is also continuous.

Exercise 4 (2 + 4 points)

Consider the function $\Phi_{\texttt{fact}} : \langle \mathbb{Z}_\perp \to \mathbb{Z}_\perp \rangle \to \langle \mathbb{Z}_\perp \to \mathbb{Z}_\perp \rangle$ defined as follows ($\Phi_{\texttt{fact}} \cong \mathit{ff}$ from the lecture):

$$(\Phi_{\texttt{fact}}(g))(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ g(x-1) * x, & \text{otherwise} \end{cases}$$

Here, $-$ and $*$ are the strict extensions of the subtraction and the multiplication function.

Show that

(a) $\Phi_{\texttt{fact}}$ is monotonic.

(b) $\Phi_{\texttt{fact}}$ is continuous.

Hint: Use the lemmas from exercise 1 of this sheet and exercise 7 of sheet 6.

Exercise 5 (1 + 2 + 1 points)

Consider the following functions:

```
fact :: Int -> Int
fact = \ x -> if x<=0 then 1 else fact(x-1) * x

one :: Int -> Int
one =  \ x -> 1

plus :: (Int,Int) -> Int
plus = \ (x,y) -> if x<=0 then y else 1 + plus (x-1,y)

inf :: Int -> Int
inf =  \ x -> inf x
```

The semantics of `fact` is defined as the lfp of $\Phi_{\texttt{fact}}$ (see exercise 4). For the functions `one`, `plus`, and `inf` give the corresponding higher-order functions $\Phi_{\texttt{one}}$, $\Phi_{\texttt{plus}}$, and $\Phi_{\texttt{inf}}$. (Here, Φ_f corresponds to f in the same way as $\Phi_{\texttt{fact}}$ corresponds to `fact`. In other words, the semantics of f is defined as the lfp of Φ_f.) Give all their fixpoints and determine the least one. What does the function $\Phi_f^n(\bot)$ compute for $n \in \mathbb{N}$, $f \in \{\texttt{one}, \texttt{plus}, \texttt{inf}\}$ (where $\Phi_f^n(\bot)$ denotes n applications of Φ_f to $\bot$)?