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Exercises *Functional Programming* – Sheet 6

Solutions will be collected until Wednesday, May 23, 2007 in the exercise course.
Exercises can be solved both in English and in German.

Exercise 1 (2 + 1 points)

We consider $D = 2^{\mathbb{Z}}$ and the order $\supseteq$ on D, i.e., D contains all sets of integers and $\supseteq$ is the superset relation. For example, D contains $\{-1, 0, 1\}$ as well as $\{1, 2, 3, 4, \ldots\}$ and $\{1, 2, 3\} \supseteq \{1, 2\}$. Note that for this relation, $\{1, 2, 3\}$ is smaller than $\{1, 2\}$. Indeed, the empty set $\emptyset$ is the *largest* element in this domain.

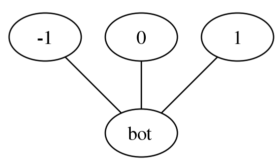
(a) What is the lub of a chain $\{M_1, M_2, \ldots\}$ with $M_i \in D$ for all i?
Show that $\supseteq$ is a complete order on D.

(b) Give an example of an infinite chain in $\supseteq_D$.

Exercise 2 (2+1 points)

Let $\mathbb{Z}_3$ be the subset $\{-1, 0, 1\}$ of $\mathbb{Z}$.

(a) The order $\sqsubseteq_{\mathbb{Z}_{3\bot}}$ can be represented graphically like this:



Present the order $\sqsubseteq_{\mathbb{B} \to \mathbb{Z}_{3\bot}}$ graphically. How many elements does it have? How many elements are maximal? Here, an element d is called maximal if and only if there is no $d' \neq d$ with $d \sqsubseteq d'$.

(b) How many elements does the order $\sqsubseteq_{\mathbb{B}_\bot \to \mathbb{Z}_{3\bot}}$ have?

Exercise 3 (1+2+2 points)

Let $\sqsubseteq_D$ and $\sqsubseteq_E$ be orders on D and E respectively. A function $f : D \to E$ is said to be *monotonic* if and only if $d \sqsubseteq_D d'$ implies $f(d) \sqsubseteq_E f(d')$.

Find all monotonic extensions of the following functions (given their usual interpretation):

(a) $\neg : \mathbb{B} \to \mathbb{B}_\perp$

(b) $\wedge, \vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}_\perp$

(c) $-, / : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_\perp$
where for all $x \in \mathbb{Z}$, x/y is $\perp_{\mathbb{Z}_\perp}$ for $y = 0$ and $\lfloor \frac{x}{y} \rfloor$ otherwise.

Exercise 4 (1+2 points)

Consider the function `times` defined as follows:

```
times :: (Int, Int) -> Int
times (0, y) = 0
times (x, y) = y + (times (x-1, y))
```

The semantics of `times` is the function $f : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$ with

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0 \\ x * y & \text{if } x > 0 \ \wedge \ y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

Prove or disprove the following statements.

- The function f is strict.
- The function f is monotonic.

Exercise 5 (3 points)

Let $\sqsubseteq_{D_1}, \ldots, \sqsubseteq_{D_n}, \sqsubseteq_E$ be orders. A function $g : D_1 \times \ldots \times D_n \to E$ is called *monotonic in the i-th argument* if and only if $d_i, d'_i \in D_i$ with $d_i \sqsubseteq_{D_i} d'_i$ implies

$$g(d_1, ..., d_{i-1}, d_i, d_{i+1}, ..., d_n) \sqsubseteq_E g(d_1, ..., d_{i-1}, d'{}_i, d_{i+1}, ..., d_n)$$

Show that g is monotonic in all arguments $1 \leq i \leq n$ if and only if it is monotonic, i.e., $(d_1, \ldots, d_n) \sqsubseteq_{D_1 \times \ldots \times D_n} (d'_1, \ldots, d'_n)$ implies $g(d_1, \ldots, d_n) \sqsubseteq_E g(d'_1, \ldots, d'_n)$.