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Exercises *Functional Programming* – Sheet 7

Solutions will be collected until Wednesday, June 6, 2007 in the exercise course.
Exercises can be solved both in English and in German.

Exercise 1 (2 points)

Let D_1, D_2, and D_3 be domains with complete orders $\sqsubseteq_1, \sqsubseteq_2, \sqsubseteq_3$, respectively. Let $f : D_1 \longrightarrow D_2$ and $g : D_2 \longrightarrow D_3$ be continuous functions. Show that the composition $g \circ f : D_1 \longrightarrow D_3$ is also continuous.

Exercise 2 (2 + 4 points)

Consider the following Haskell declarations for `sum` and the associated higher-order function `f_sum`:

```
sum   :: Int -> Int
sum   = \x -> if x <= 0 then 0 else sum (x-1) + x

f_sum = \g -> \x -> if x <= 0 then 0 else g (x-1) + x
```

The function $\Phi_{\texttt{f\_sum}} : \langle \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \rangle \rightarrow \langle \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \rangle$ is defined as follows:

$$(\Phi_{\texttt{f\_sum}}(g))(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ g(x-1) + x, & \text{otherwise} \end{cases}$$

Here, $-$ and $+$ are the strict extensions of the subtraction and the multiplication function.

Show that

(a) $\Phi_{\texttt{f\_sum}}$ is monotonic.

(b) $\Phi_{\texttt{f\_sum}}$ is continuous.

Hint: You should use Lemma 2.1.11 and Theorem 2.1.15 in your proof.

Exercise 3 (1 + 1 + 2 + 3 points)

Consider the following Haskell functions:

```
fact  :: Int -> Int
fact  = \x -> if x <= 0 then 1 else fact (x-1) * x

three :: Int -> Int
three = \x -> 3

incr  :: Int -> Int
incr  = \x -> incr (x+3)

times :: (Int,Int) -> Int
times = \(x,y) -> if y <= 0 then 0 else x + times (x,y-1)
```

The higher-order Haskell function `f_fact` corresponding to `fact` is:

```
f_fact = \g -> \x -> if x <= 0 then 1 else g (x-1) * x
```

The semantics $\phi_{\texttt{f\_fact}}$ of `f_fact` is:

$$(\phi_{\texttt{f\_fact}}(g))(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ g(x-1) \cdot x, & \text{otherwise} \end{cases}$$

The semantics $\phi_{\texttt{fact}}$ of `fact` is the least fixpoint of $\phi_{\texttt{f\_fact}}$ (where for all $x < 0$ we define $x! = 1$):

$$\phi_{\texttt{fact}}(x) = \begin{cases} x!, & \text{if } x \in \mathbb{Z} \\ \bot & \text{otherwise} \end{cases}$$

(a) Give the Haskell definitions for the higher-order functions `f_three`, `f_incr`, and `f_times` corresponding to `three`, `incr`, and `times`.

(b) Give the semantics $\phi_{\texttt{f\_three}}$, $\phi_{\texttt{f\_incr}}$, and $\phi_{\texttt{f\_times}}$ of the functions `f_three`, `f_incr`, and `f_times`.

(c) What does the function $\phi_f^n(\bot)$ compute for $n \in \mathbb{N}$, $f \in \{$`f_three`, `f_incr`, `f_times`$\}$?

Here, $\phi_f^n(\bot)$ denotes n applications of ϕ_f to the undefined function $\bot$.

(d) Give all fixpoints of the semantic functions $\phi_{\texttt{f\_three}}$, $\phi_{\texttt{f\_incr}}$, and $\phi_{\texttt{f\_times}}$ from (a). Which ones are the least fixpoints (i.e, which correspond to $\phi_{\texttt{three}}$, $\phi_{\texttt{incr}}$, and $\phi_{\texttt{times}}$ respectively)?