First name	Last name	Matriculation number

Exercise 1 (4 + 3 + 4 + 6 + 5 = 22 points)

The following data structure represents polymorphic lists that can contain values of *two* types in arbitrary order:

```
data DuoList a b = C a (DuoList a b) | D b (DuoList a b) | E
```

Consider the following list `zs` of integers and characters:

$$[\,4,\ 'a',\ 'b',\ 6\,]$$

The representation of `zs` as an object of type `DuoList Int Char` in Haskell would be:

```
C 4 (D 'a' (D 'b' (C 6 E)))
```

Implement the following functions in Haskell.

(a) The function `foldDuo` of type

```
(a -> c -> c) -> (b -> c -> c) -> c -> DuoList a b -> c
```

works as follows: `foldDuo f g h xs` replaces all occurrences of the constructor `C` in the list `xs` by `f`, it replaces all occurrences of the constructor `D` in `xs` by `g`, and it replaces all occurrences of the constructor `E` in `xs` by `h`. So for the list `zs` above,

```
foldDuo (*) (\x y -> y) 3 zs
```

should compute

```
(*) 4 ((\x y -> y) 'a' ((\x y -> y) 'b' ((*) 6 3))),
```

which in the end results in `72`. Here, `C` is replaced by `(*)`, `D` is replaced by `(\x y -> y)`, and `E` is replaced by `3`.

```
foldDuo f g h (C x xs)    = f x (foldDuo f g h xs)
foldDuo f g h (D x xs)    = g x (foldDuo f g h xs)
foldDuo f g h E           = h
```

First name	Last name	Matriculation number

(b) Use the `foldDuo` function from (a) to implement the `cd` function which has the type `DuoList Int a -> Int` and returns the sum of the *entries* under the data constructor `C` and of the *number of elements* built with the data constructor `D`.

In our example above, the call `cd zs` should have the result `12`. The reason is that `zs` contains the entries `4` and `6` under the constructor `C` and it contains two elements `'a'` and `'b'` built with the data constructor `D`.

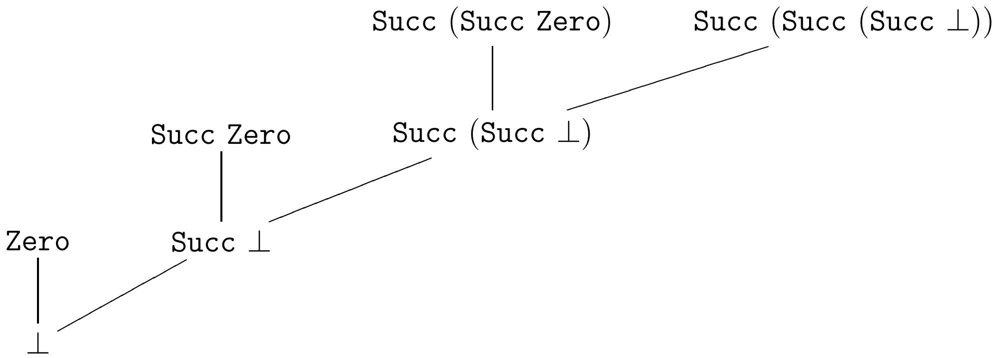
```
cd = foldDuo (+) (\x y -> y + 1) 0
```

First name	Last name	Matriculation number

(c) Consider the following data type declaration for natural numbers:

```
data Nats = Zero | Succ Nats
```

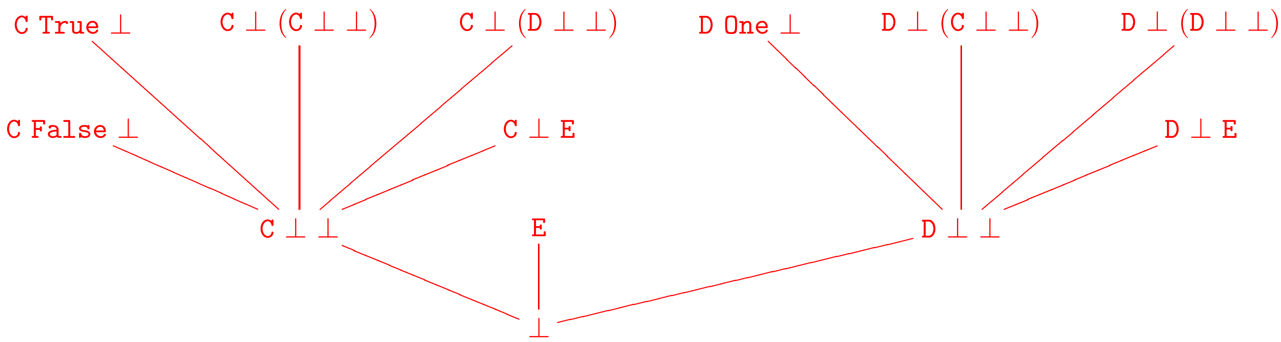
A graphical representation of the first four levels of the domain for `Nats` could look like this:



We define the following data type `Single`, which has only one data constructor `One`:

```
data Single = One
```

Sketch a graphical representation of the first three levels of the domain for the data type `DuoList Bool Single`.



First name	Last name	Matriculation number

(d) The *digit sum* of a natural number is the sum of all digits of its decimal representation. For example, the digit sum of the number 6042 is $6 + 0 + 4 + 2 = 12$. Write a Haskell function `digitSum :: Int -> Int` that takes a natural number and returns its digit sum. Your function may behave arbitrarily on negative numbers. It can be helpful to use the pre-defined functions `div, mod :: Int -> Int -> Int` to compute result and remainder of division, respectively. For example, `div 7 3` is `2` and `mod 7 3` is `1`.

```
digitSum :: Int -> Int
digitSum 0 = 0
digitSum (n+1) = mod (n+1) 10 + digitSum (div (n+1) 10)
```

Now implement a function `digitSumList :: Int -> Int -> [Int]` where `digitSumList n b` returns a list of all those numbers `x` where $0 \leq$ `x` $\leq$ `b` and where the digit sum of `x` is `n`. Perform your implementation only with the help of a **list comprehension**, i.e., you should use exactly one declaration of the following form:

```
digitSumList ... = [ ... | ... ]
```

Of course, here you can (and should) make use of the function `digitSum` to compute the digit sum of a number.

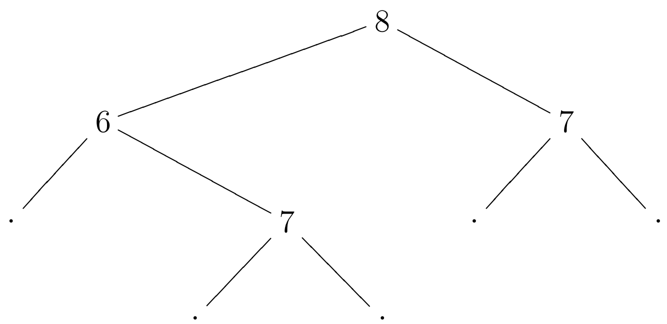
```
digitSumList :: Int -> Int -> [Int]
digitSumList n b = [ x | x <- [0..b], digitSum x == n ]
```

First name	Last name	Matriculation number

(e) The following data structure represents binary trees only containing values in the inner nodes:

```
data Tree a = Leaf | Node a (Tree a) (Tree a)
```

Consider the following tree `t` of integers:



The representation of `t` as an object of type `Tree Int` in Haskell would be:

```
t = Node 8 (Node 6 Leaf (Node 7 Leaf Leaf)) (Node 7 Leaf Leaf)
```

We define the *fringe* of a tree to be those nodes that have two leaves as children. Write a Haskell function `fringe :: Tree a -> [a]` which computes a list of all the values in the nodes of the fringe (with repetition, i.e., a value should appear in the result list as many times as it appears in a fringe node). As an example, `fringe t` should return `[7,7]`.

```
fringe Leaf = []
fringe (Node a Leaf Leaf) = [a]
fringe (Node a t1 t2) = (fringe t1) ++ (fringe t2)
```

First name	Last name	Matriculation number

Exercise 2 (4 + 5 = 9 points)

Consider the following Haskell declarations for the `square` function:

```
square :: Int -> Int
square 0     = 0
square (x+1) = 1 + 2*x + square x
```

(a) Please give the Haskell declarations for the higher-order function `f_square` corresponding to `square`, i.e., the higher-order function `f_square` such that the least fixpoint of `f_square` is `square`. In addition to the function declaration(s), please also give the type declaration of `f_square`. Since you may use full Haskell for `f_square`, you do not need to translate `square` into simple Haskell.

```
f_square :: (Int -> Int) -> (Int -> Int)
f_square square 0 = 0
f_square square (x+1) = 1 + 2*x + square x
```

(b) We add the Haskell declaration `bot = bot`. For each $n \in \mathbb{N}$ please determine which function is computed by $\texttt{f\_square}^n$ `bot`. Here "$\texttt{f\_square}^n$ `bot`" represents the n-fold application of `f_square` to `bot`, i.e., it is short for $\underbrace{\texttt{f\_square (f\_square} \ldots \texttt{(f\_square}}_{n \text{ times}}\ \texttt{bot)}\ldots)$.

Let $f_n : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$ be the function that is computed by $\texttt{f\_square}^n$ `bot`.
Give f_n in **closed form**, i.e., using a non-recursive definition.

$$(\texttt{f\_square}^n(\perp))(x) = \begin{cases} x^2, & \text{if } 0 \leq x < n \\ \perp, & \text{otherwise} \end{cases}$$

First name	Last name	Matriculation number

Exercise 3 (6 points)

Let D_1, D_2 be domains, let $\sqsubseteq_{D_2}$ be a complete partial order on D_2. As we know from the lecture, then also $\sqsubseteq_{D_1 \to D_2}$ is a complete partial order on the set of all functions from D_1 to D_2.

Prove that $\sqsubseteq_{D_1 \to D_2}$ is also a complete partial order on the set of all *constant* functions from D_1 to D_2. A function $f : D_1 \to D_2$ is called *constant* iff $f(x) = f(y)$ holds for all $x, y \in D_1$.

> *Hint:* The following lemma may be helpful:
>
> If S is a chain of functions from D_1 to D_2, then $\sqcup S$ is the function with:
>
> $$(\sqcup S)(x) = \sqcup\{f(x) \mid f \in S\}$$

We need to show two statements:

a) The set of all constant functions from D_1 to D_2 has a smallest element $\bot$.

   Obviously, the constant function f with $f(x) = \bot$ for all $x \in D_1$ satisfies this requirement.

b) For every chain S on the set of all constant functions from D_1 to D_2 there is a least upper bound $\sqcup S$ which is an element of the set of all constant functions from D_1 to D_2.

   Let S be a chain of constant functions from D_1 to D_2. By the above lemma, we have $(\sqcup S)(x) = \sqcup\{f(x) \mid f \in S\}$. It remains to show that the function $\sqcup S : D_1 \to D_2$ actually is a constant function. For all $x, y \in D_1$, we have:

$$\begin{array}{cll} & (\sqcup S)(x) & \\ = & \sqcup\{f(x) \mid f \in S\} & \\ = & \sqcup\{f(y) \mid f \in S\} & \text{since the elements of } S \text{ are constant functions} \\ = & (\sqcup S)(y) & \end{array}$$

   Therefore, also $(\sqcup S)(x)$ is a constant function.

□

First name	Last name	Matriculation number

Exercise 4 (4 + 5 = 9 points)

Consider the following data structure for polymorphic lists:

```
data List a = Nil | Cons a (List a)
```

(a) Please translate the following Haskell-expression into an equivalent lambda term (e.g., using $\mathcal{L}am$). Recall that pre-defined functions like `even` are translated into constants of the lambda calculus.

It suffices to give the result of the transformation.

```
let f = \x -> if (even x) then Nil else Cons x (f x)
    in f
```

$(\texttt{fix}\ (\lambda f\ x.\ \texttt{if}\ (\texttt{even}\ x)\ \texttt{Nil}\ (\texttt{Cons}\ x\ (f\ x))\ )\ )$

First name	Last name	Matriculation number

(b) Let δ be the set of rules for evaluating the lambda terms resulting from Haskell, i.e., δ contains at least the following rules:

$$\begin{aligned} \texttt{fix} &\rightarrow \lambda f.\ f\ (\texttt{fix}\ f) \\ \texttt{plus}\ 2\ 3 &\rightarrow 5 \end{aligned}$$

Now let the lambda term t be defined as follows:

$$t = (\texttt{fix}\ (\lambda g\ x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil}))\ 2$$

Please reduce the lambda term t by WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation. You have to give **all** intermediate steps until you reach **weak head normal form** (and no further steps).

$$\begin{aligned} & (\texttt{fix}\ (\lambda g\ x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil}))\ 2 \\ \rightarrow_\delta\ & ((\lambda f.\ f\ (\texttt{fix}\ f))\ (\lambda g\ x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil}))\ 2 \\ \rightarrow_\beta\ & ((\lambda g\ x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil})\ (\texttt{fix}\ (\lambda g\ x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil})))\ 2 \\ \rightarrow_\beta\ & ((\lambda x.\ \texttt{Cons}\ (\texttt{plus}\ x\ 3)\ \texttt{Nil})\ 2 \\ \rightarrow_\beta\ & \texttt{Cons}\ (\texttt{plus}\ 2\ 3)\ \texttt{Nil} \end{aligned}$$

First name	Last name	Matriculation number

10

Exercise 5 (10 points)

Use the type inference algorithm $\mathcal{W}$ to determine the most general type of the following lambda term under the initial type assumption A_0. Show the results of all sub-computations and unifications, too. If the term is not well typed, show how and why the $\mathcal{W}$-algorithm detects this.

$$\lambda f.\ (\texttt{Succ}\ (f\ x))$$

The initial type assumption A_0 contains at least the following:

$$\begin{array}{lcl} A_0(\texttt{Succ}) & = & (\texttt{Nats} \to \texttt{Nats}) \\ A_0(f) & = & \forall a.\ a \\ A_0(x) & = & \forall a.\ a \end{array}$$

$\mathcal{W}(A_0,\ \lambda f.\ (\texttt{Succ}\ (f\ x))\ )$

$\quad \mathcal{W}(A_0 + \{f :: b_1\},\ (\texttt{Succ}\ (f\ x))\ )$

$\quad\quad \mathcal{W}(A_0 + \{f :: b_1\},\ \texttt{Succ})$

$\quad\quad = (id,\ (\texttt{Nats} \to \texttt{Nats})\ )$

$\quad\quad \mathcal{W}(A_0 + \{f :: b_1\},\ (f\ x)\ )$

$\quad\quad\quad \mathcal{W}(A_0 + \{f :: b_1\},\ f)$

$\quad\quad\quad = (id,\ b_1)$

$\quad\quad\quad \mathcal{W}(A_0 + \{f :: b_1\},\ x)$

$\quad\quad\quad = (id,\ b_2)$

$\quad\quad\quad mgu(b_1,\ (b_2 \to b_3)\ ) = [b_1/(b_2 \to b_3)]$

$\quad\quad = ([b_1/(b_2 \to b_3)],\ b_3)$

$\quad\quad mgu((\texttt{Nats} \to \texttt{Nats}),\ (b_3 \to b_4)\ ) = [b_3/\texttt{Nats}, b_4/\texttt{Nats}]$

$\quad = ([b_1/(b_2 \to \texttt{Nats}), b_3/\texttt{Nats}, b_4/\texttt{Nats}],\ \texttt{Nats})$

$= ([b_1/(b_2 \to \texttt{Nats}), b_3/\texttt{Nats}, b_4/\texttt{Nats}],\ ((b_2 \to \texttt{Nats}) \to \texttt{Nats})\ )$

Resulting type: $((b_2 \to \texttt{Nats}) \to \texttt{Nats})$