Exercise 1 (Quiz): (4 + 5 + 5 + 3 + 3 = 20 points)

Give a short proof sketch or a counterexample for each of the following statements:

a) Monotonic unary functions are always strict.

b) Strict unary functions on flat domains are always monotonic.

c) Let $\mathbb{B}$ be the Boolean values `true`, `false`.
Is $f : (\mathbb{B} \to \mathbb{B}_\perp) \to \mathbb{Z}$ with $f(g) = \begin{cases} 1 & \text{if } g(x) \neq \texttt{true} \text{ for all } x \in \mathbb{B} \\ 0 & \text{otherwise} \end{cases}$ monotonic?

d) Is $\to_\alpha$ terminating?

e) Is $\to_\alpha$ confluent?

Solution:

a) **No**. Consider $f(x) = 0$.

b) **Yes**. Let $f : D- > D'$ be some strict function, D flat, $d, d' \in D$ and $d \sqsubseteq d'$. As the domain is flat, we have two cases:

- $d = d'$, which directly implies $f(d) = f(d')$.
- $d = \perp$, which due to f's strictness implies $f(d) = \perp \sqsubseteq f(d')$.

c) **No**. Consider $g_1(x) = \perp$ and $g_2(x) = \begin{cases} \texttt{true} & \text{if } x = \texttt{true} \\ \perp & \text{otherwise} \end{cases}$. We have $g_1 \sqsubseteq g_2$, but $f(g_1) = 1 \not\sqsubseteq 0 = f(g_2)$.

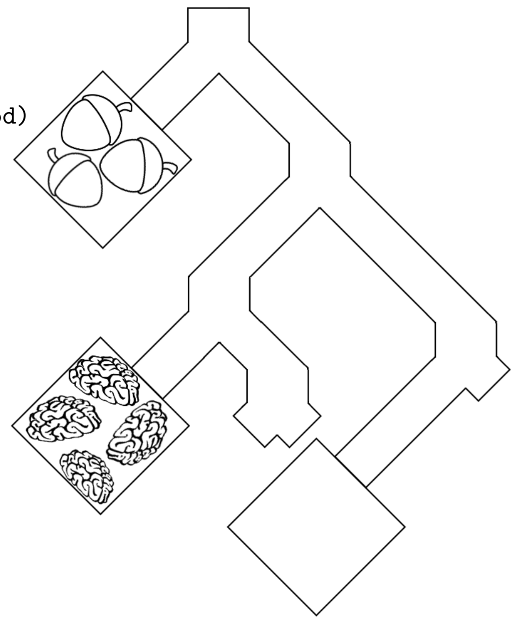
d) **No**. Consider the derivation $\lambda x.x \to_\alpha \lambda y.y \to_\alpha \lambda x.x$ as counterexample.

e) **Yes**. Let t be some λ-term and $t \to_\alpha^* q$, $t \to_\alpha^* p$ hold. Then, as q and p are just variable-renamed versions of t, we can directly conclude that $q \to_\alpha^* p$ and $p \to_\alpha^* q$ holds.

Exercise 2 (Programming in Haskell): (8 + 10 + 10 + 6 + 8 = 42 points)

We define a polymorphic data structure `ZombieHalls` to represent a zombie-infested school whose classrooms contain different types of food:

```
data ZombieHalls food =
    HallwayFork (ZombieHalls food) (ZombieHalls food)
  | HallwayClassroom (Int, food) (ZombieHalls food)
  | HallwayEnd
```



Here, we use three data constructors: One representing the case that the hallway forks and we can go in two directions, one for the case that we have a classroom on one side and can continue in the hallway and finally one case for the end of a hallway. The data structure `ZombieFood` is used to represent food for zombies. As example, consider the following definition of `exampleSchool` of type `ZombieLabyrinth ZombieFood`, corresponding to the illustration on the right:

```
data ZombieFood = Brains | Nuts deriving Show

exampleSchool :: ZombieHalls ZombieFood
exampleSchool =
    HallwayClassroom (3, Nuts)
     (HallwayFork
       (HallwayClassroom (4, Brains)
         (HallwayFork HallwayEnd HallwayEnd))
       (HallwayClassroom (0, Brains) HallwayEnd))
```

a) Implement a function `buildSchool :: Int -> ZombieHalls ZombieFood` such that for any integer number `n` ≥ 0, it returns a structure of hallways containing 2^{n+1} classrooms in total. Of these, one half should each contain one brain and the other should each contain one nut.

b) Implement a fold function `foldZombieHalls`, including its type declaration, for the data structure `ZombieHalls`. As usual, the fold function replaces the data constructors in a `ZombieHalls` expression by functions specified by the user. The first argument of `foldZombieHalls` should be the function for the case of a `HallwayFork`, the second argument should replace the `HallwayClassroom` constructor and the third argument should replace the `HallwayEnd` data constructor. As an example, consider the following function definition, which uses `foldZombieHalls` to determine the number of dead ends in a `ZombieHalls` structure, where a classroom does not count as dead end. Hence, the call `numberOfDeadEnds exampleSchool` returns 3.

```
numberOfDeadEnds :: ZombieHalls food -> Int
numberOfDeadEnds school = foldZombieHalls (+) (\_ r -> r) 1 school
```

c) Implement the function `bcCounter :: ZombieHalls ZombieFood -> (Int, Int)`, which counts the number of **b**rains and **c**lassrooms in a given school and returns the two numbers as a tuple of integers. The first part of the tuple should be the number of brains in the school and the second should be the number of classrooms. For the definition of `bcCounter`, use only one defining equation where the right-hand side **is just one call to the function** `foldZombieHalls`. However, you may use and define non-recursive auxiliary functions.

For example, a call `bcCounter exampleSchool` should return the tuple `(4, 3)`.

d) The infinite sequence of Fibonacci numbers fib_i is defined as $fib_0 = 0$, $fib_1 = 1$ and $fib_i = fib_{i-1} + fib_{i-2}$ for all $i > 1$. The first elements of the sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

Implement a cyclic data structure `fibs :: [Int]` that represents the infinite list of Fibonacci numbers. Do not use self-defined auxiliary functions and ensure that `take n fibs` has *linear* complexity.

Hints:

- You should use use the function `zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]`, which applies the function given as its first argument to combine the elements of two lists. For example `zipWith (++) ["a","b"] ["c", "d", "e"]` results in the list `["ac","bd"]`. Note that the length of the resulting list is the smallest length of both input lists.
- You may use the pre-defined function `tail` defined as `tail (x:xs) = xs`.

e) Write a function `splits :: [a] -> [([a],[a])]` that computes all *splits* of a finite input list, i.e., a call `splits xs` should return all pairs `(ys,zs)` such that `ys ++ zs` is again `xs`. For example, we have `splits "abc" = [("","abc"),("a","bc"),("ab","c"),("abc","")]`.

The right-hand side of your function should be **just a list comprehension**.

Hints:

- Use `length :: [a] -> Int`, wich returns the length of a given list.
- Use `take :: Int -> [a] -> [a]`, where `take n xs` yields the longest prefix of `xs` with length $\leq$ `n`.
- Use `drop :: Int -> [a] -> [a]`, where `drop n xs` returns the list obtained from `xs` by removing the first `n` elements.

Solution:

a)
```
buildSchool :: Int -> ZombieHalls ZombieFood
buildSchool n | n == 0 = HallwayClassroom (1, Nuts) (HallwayClassroom (1, Brains) HallwayEnd)
              | n > 0  = HallwayFork otherHall otherHall
  where otherHall = buildSchool (n-1)
```

b)
```
foldZombieHalls
  :: (result -> result -> result)
  -> ((Int, food) -> result -> result)
  -> result
  -> ZombieHalls food
  -> result
foldZombieHalls fHF fHC fHE = go
  where
    go (HallwayFork l r) = fHF (go l) (go r)
    go (HallwayClassroom c h) = fHC c (go h)
    go HallwayEnd = fHE
```

c)
```
bcCounter :: ZombieHalls ZombieFood -> (Int, Int)
bcCounter = foldZombieHalls (\(rB, rN) (cB, cN) -> (rB+cB, rN+cN)) cHelper (0,0)
 where
  cHelper (n, Brains) (rB, rN) = (rB+n, rN+1)
  cHelper (n, _) (rB, rN) = (rB, rN+1)
```

d)
```
fibs :: [Int]
fibs = 0:1:(zipWith (+) fibs (tail fibs))
```

e)
```
splits :: [a] -> [([a],[a])]
splits xs = [ (take i xs, drop i xs) | i <- [0 .. length xs] ]
```

Exercise 3 (Semantics): (22 + 10 + 5 + 4 = 41 points)

a) i) Let $\sqsubseteq_{D_1}$ and $\sqsubseteq_{D_2}$ be complete partial orders on D_1 resp. D_2 and $f : D_1 \to D_2$ a function. Prove that f is continuous if and only if f is monotonic and for all chains S in D_1, $f(\sqcup S) \sqsubseteq_{D_2} \sqcup f(S)$ holds.

ii) Let $D = \mathbb{N} \to \{1\}_\perp$, i.e., D is the set of all functions mapping the natural numbers to $\perp$ or 1. Let $\sqsubseteq$ be defined as usual on functions.

1) Prove that every chain $S \sqsubseteq D$ has a least upper bound w.r.t. the relation $\sqsubseteq$.

2) Prove that $\sqsubseteq$ is a cpo on D.

3) Give an example for an infinite chain in $(D, \sqsubseteq)$.

4) Give a monotonic, non-continuous function $f : D \to D$. You do not need to prove that f has these properties.

b) i) Consider the following Haskell function `exp`:

```
exp :: (Int, Int) -> Int
exp (x, 0) = 1
exp (x, y) = x * exp (x, y - 1)
```

Please give the Haskell declaration for the higher-order function `f_exp` corresponding to `exp`, i.e., the higher-order function `f_exp` such that the least fixpoint of `f_exp` is `exp`. In addition to the function declaration, please also give the type declaration of `f_exp`. You may use full Haskell for `f_exp`.

ii) Let $\phi_{\texttt{f\_exp}}$ be the semantics of the function `f_exp`. Give the semantics of $\phi^n_{\texttt{f\_exp}}(\bot)$ for $n \in \mathbb{N}$, i.e., the semantics of the n-fold application of $\phi_{\texttt{f\_exp}}$ to $\bot$.

iii) Give the least fixpoint of $\phi_{\texttt{f\_exp}}$.

c) Consider the following data type declaration for natural numbers:

```
data Nats = Z | S Nats
```

A graphical representation of the first four levels of the domain for `Nats` could look like this:

$$\begin{tikzcd}
 & & \texttt{S (S Z)} & & \texttt{S (S (S }\bot\texttt{))} \\
 & \texttt{S Z} & & \texttt{S (S }\bot\texttt{)} \arrow[ul] \arrow[ur] & \\
\texttt{Z} & & \texttt{S }\bot \arrow[ul] \arrow[ur] & & \\
 & \bot \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Now consider the following data type declarations:

```
data U = V
data T a = C | D (T a) | E a a
```

Give a graphical representation of the first three levels of the domain for the type `T U`. The third level contains the element `D C`, for example.

d) Consider the usual definitions for `List a`, i.e., `data List a = Nil | Cons a (List a)` and `Nats` from above.

Write a function `length :: List a -> Nats` in **Simple** Haskell that computes the length of a list, i.e., `length (Cons Z (Cons Z Nil))` should yield `S(S(Z))`. Your solution should use the functions defined in the transformation from the lecture such as $\texttt{sel}_{n,i}$, $\texttt{isa}_{\underline{\texttt{constr}}}$, $\texttt{argof}_{\underline{\texttt{constr}}}$, and `bot`. You do not have to use the transformation rules from the lecture, though.

Solution:

a) i) First, let f be continuous. Then, for any chain S, we have $f(\sqcup S) = \sqcup f(S)$. Because $\sqsubseteq_{D_2}$ is reflexive, $f(\sqcup S) \sqsubseteq_{D_2} \sqcup f(S)$ follows. To prove monotonicity of f, let $d, d' \in D_1$ with $d \sqsubseteq_{D_1} d'$. Then $f(\sqcup\{d, d'\}) = f(d')$. Since f is continuous, we also have $f(\sqcup\{d, d'\}) = \sqcup\{f(d), f(d')\}$. Consequently, we have $f(d) \sqsubseteq f(d')$.

Now, assume f is monotonic and $f(\sqcup S) \sqsubseteq_{D_2} \sqcup f(S)$ holds. As $\sqsubseteq_{D_2}$ is antisymmetric, it suffices to prove $\sqcup f(S) \sqsubseteq_{D_2} f(\sqcup S)$, i.e., that for all $d \in S$, we have $f(d) \sqsubseteq_{D_2} f(\sqcup S)$. Obviously, $d \sqsubseteq_{D_1} \sqcup S$ holds. As f is monotonic, $f(d) \sqsubseteq f(\sqcup S)$ follows and therefore, $\sqcup f(S) \sqsubseteq_{D_2} f(\sqcup S)$ holds.

ii) Let $S = \{f_1, f_2, \dots\}$ with $f_i \sqsubseteq f_{i+1}$ be a chain

1) Let $M_i = \{x \in \mathbb{N} \mid f(x) \neq \bot\}$. Then, by definition of $\sqsubseteq$, $M_i \subseteq M_{i+1}$. Let $M = \bigcup M_i$. We define $\sqcup S = f$ with $f(x) = \begin{cases} 1 & \text{if } x \in M \\ \bot & \text{otherwise} \end{cases}$.

First, we prove that f is an upper bound. Assume $f_i \sqsubseteq f$ does not hold. Then, there is some $n \in \mathbb{N}$ with $f_i(n) = 1$ and $f(n) = \bot$. But then, we also have $n \in M_i$ and hence $n \in M$, so $f(n) = f_i(n) = 1$, which is a contradiction to our choice of n.

Now, we prove that f is the least upper bound. Assume there is another bound $g \neq f$ with $g \sqsubseteq f$. Then, there is some $n \in \mathbb{N}$ with $g(n) = \bot$ and $f(n) = 1$. But then, there is also some k such that $n \in M_k$, i.e., $f_k(n) = 1$, so $f_k(n) \not\sqsubseteq g(n)$ and hence, g is not an upper bound for S.

2) In 1), we have proven that for every chain, there exists a lub. The constructed function is trivially again in D. We also have $c(x) = \bot \in D$ as obvious minimal element, hence, $\sqsubseteq$ is a cpo for D.

3) Let $N_i := \{k \in \mathbb{N} \mid k \leq i\}$. Then, $N_i \subseteq N_{i+1}$ holds. Let

$$f_i(x) = \begin{cases} 1 & \text{if } x \in N_i \\ \bot & \text{otherwise} \end{cases}$$

Then, $f_i \sqsubseteq f_{i+1}$ holds (see above) and hence, $\{f_1, f_2, \dots\}$ is a chain.

4)

$$f(g) = \begin{cases} h(y) = \bot & \text{if } \{x \in \mathbb{N} \mid g(x) \neq \bot\} \text{ is finite} \\ h(y) = 1 & \text{otherwise} \end{cases}$$

b) i)

```
f_exp :: ((Int, Int) -> Int) -> ((Int, Int) -> Int)
f_exp exp (x, 0) = 1
f_exp exp (x, y) = x * exp (x, y - 1)
```

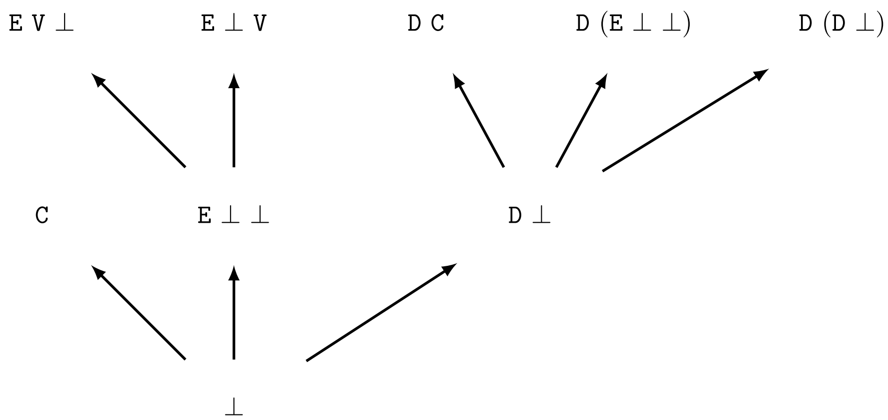
ii)

$$(\phi^n_{\texttt{f\_exp}}(\bot))(x,y) = \begin{cases} 1 & \text{if } y = 0 \land n > 0 \\ x^y & 0 < y < n \land x \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

iii) The least fixpoint of $\phi_{\texttt{f\_exp}}$ is the function

$$g(x,y) = \begin{cases} 1 & \text{if } y = 0 \\ x^y & 0 < y \land x \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

c)



d)
```
length = \xs ->
           if (isa_Nil xs) then Z
           else if (isa_Cons xs)
                then S(length (sel_2,2 (argof_Cons xs)))
                else bot
```

Alternative:

```
length = \xs ->
           if (isa_Nil xs) then Z
           else S(length (sel_2,2 (argof_Cons xs)))
```

Exercise 4 (Lambda Calculus): (4 + 6 = 10 points)

a) Please translate the following Haskell expression into an equivalent lambda term (e.g., using $\mathcal{L}am$). Translate the pre-defined function `>` to `GreaterThan`, `+` to `Plus`, `*` to `Times` and `-` to `Minus` (remember that the infix notation of `>`, `+`, `*`, `-` is not allowed in lambda calculus). It suffices to give the result of the transformation:

```
let sqrt = \x a -> if a * a > x then a - 1 else sqrt x (a + 1) in sqrt u 0
```

b) Let $t = \lambda fromto.\lambda x.\lambda y.\texttt{If}\ (\texttt{Eq}\ x\ y)\ \texttt{Nil}\ (\texttt{Cons}\ x\ (fromto\ (\texttt{Plus}\ x\ 1)\ y))$ and

$$\begin{aligned}\delta = \ & \{\ \texttt{If True} \to \lambda x.\lambda y.x,\\ & \quad \texttt{If False} \to \lambda x.\lambda y.y,\\ & \quad \texttt{Fix} \to \lambda f.f\ (\texttt{Fix}\ f)\}\\ \cup & \ \{\ \texttt{Plus}\ x\ y \to z \mid x,y \in \mathbb{Z} \wedge z = x+y\}\\ \cup & \ \{\ \texttt{Eq}\ x\ y \to \texttt{False} \mid x,y \in \mathbb{Z} \wedge x \neq y\}\\ \cup & \ \{\ \texttt{Eq}\ x\ y \to \texttt{True} \mid x,y \in \mathbb{Z} \wedge x = y\}\end{aligned}$$

Please reduce $\texttt{Fix}\ t\ 1\ 2$ by WHNO-reduction with the $\to_{\beta\delta}$-relation. List **all** intermediate steps until reaching weak head normal form, but please write "t" instead of the term it represents whenever possible. However, you may combine several subsequent $\to_\beta$-steps.

Solution:

a) $(\texttt{Fix}\ (\lambda sqrt\ x\ a.\texttt{If}\ (\texttt{Greater}\ (\texttt{Times}\ a\ a)\ x)\ (\texttt{Minus}\ a\ 1)\ (sqrt\ x\ (\texttt{Plus}\ a\ 1))))\ u\ 0$

b)

$$\begin{aligned}& \texttt{Fix}\ t\ 1\ 2\\ \to_\delta\ & (\lambda f.(f\ (\texttt{Fix}\ f)))\ t\ 1\ 2\\ \to_\beta\ & t\ (\texttt{Fix}\ t)\ 1\ 2\\ \to_\beta\ & (\lambda x.\lambda y.\texttt{If}\ (\texttt{Eq}\ x\ y)\ \texttt{Nil}\ (\texttt{Cons}\ x\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ x\ 1)\ y)))\ 1\ 2\\ \to_\beta\ & (\lambda y.\texttt{If}\ (\texttt{Eq}\ 1\ y)\ \texttt{Nil}\ (\texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ y)))\ 2\\ \to_\beta\ & \texttt{If}\ (\texttt{Eq}\ 1\ 2)\ \texttt{Nil}\ (\texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ 2))\\ \to_\delta\ & \texttt{If False Nil}\ (\texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ 2))\\ \to_\delta\ & (\lambda x.\lambda y.y)\ \texttt{Nil}\ (\texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ 2))\\ \to_\beta\ & (\lambda y.y)\ (\texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ 2))\\ \to_\beta\ & \texttt{Cons}\ 1\ ((\texttt{Fix}\ t)\ (\texttt{Plus}\ 1\ 1)\ 2)\end{aligned}$$

Exercise 5 (Type Inference): (6 points)

Using the initial type assumption $A_0 := \{y :: \forall a.a \to a\}$ infer the type of the expression $\lambda x.(y\,x)\,x$ using the algorithm $\mathcal{W}$.

Solution:

$\mathcal{W}(A_0, \lambda x.(y\,x)\,x)$
$\quad\mathcal{W}(A_0 + \{x :: b_1\}, (y\,x)\,x)$
$\quad\quad\mathcal{W}(A_0 + \{x :: b_1\}, y\,x)$
$\quad\quad\quad\mathcal{W}(A_0 + \{x :: b_1\}, y) = (id, b_2 \to b_2)$
$\quad\quad\quad\mathcal{W}(A_0 + \{x :: b_1\}, x) = (id, b_1)$
$\quad\quad mgu(b_2 \to b_2, b_1 \to b_3) = [b_1/b_2, b_3/b_2])$
$\quad\quad = ([b_1/b_2, b_3/b_2], b_2)$
$\quad\quad\mathcal{W}(A_0 + \{x :: b_2\}, x) = (id, b_2)$
$\quad mgu(b_2, b_2 \to b_4) \leadsto \textit{occur failure}$