

Inferring Lower Bounds for Runtime Complexity

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joint work with [Florian Frohn](#), [Jera Hensel](#), [Cornelius Aschermann](#), and [Thomas Ströder](#)

Runtime Complexity Analysis of TRSs

$\text{qs}(\text{nil}) \rightarrow \text{nil}$

$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$

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$$\text{qs}(\text{cons}(\text{zero}, \dots, \text{cons}(\text{zero}, \text{nil}))) \rightarrow^{3n^2+2n+1} \text{cons}(\text{zero}, \dots, \text{cons}(\text{zero}, \text{nil}))$$

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 - prove tightness of upper bounds
 - detect bugs / vulnerabilities

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 - *rewrite lemma* describes family of rewrite sequences

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$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- rewrite lemma* describes family of rewrite sequences

- Infer Bounds for TRSs

- relation between length of rewrite sequence and size of first term in sequence

- Improvements

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

Runtime Complexity Analysis of TRSs

$$\begin{array}{ll} \text{qs(nil)} \rightarrow \text{nil} & \text{qs(cons}(x, xs)\text{)} \rightarrow \text{qs}(\text{low}(x, xs)) \text{++ cons}(x, \text{qs}(\text{high}(x, xs))) \\ \text{low}(x, \text{nil}) \rightarrow \text{nil} & \text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys)) \\ \text{if(tt, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{low}(x, ys) & \text{if(ff, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{cons}(y, \text{low}(x, ys)) \end{array}$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

- Prove Conjectures

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$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

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Runtime Complexity Analysis of TRSs

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Rewrite Lemma

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1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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- Compute most general **typing** for the TRS

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- Compute most general **typing** for the TRS

`zero : Nats, succ : Nats → Nats, nil : List, cons : Nats × List → List, ...`

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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- Compute most general **typing** for the TRS

$\text{zero} : \mathbf{Nats}, \quad \text{succ} : \mathbf{Nats} \rightarrow \mathbf{Nats}, \quad \text{nil} : \mathbf{List}, \quad \text{cons} : \mathbf{Nats} \times \mathbf{List} \rightarrow \mathbf{List}, \dots$

- Generator function symbol** $\gamma_\tau : \mathbb{N} \rightarrow \tau$

1. Speculating Conjectures

 $\text{qs}(\text{nil}) \rightarrow \text{nil}$ $\text{low}(x, \text{nil}) \rightarrow \text{nil}$ $\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$ $\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$ $\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$ $\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$ γ_{Nats} γ_{List}

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- to express families of terms of type τ

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γ_{Nats}

γ_{List}

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- to express families of terms of type τ
- $\gamma_\tau(n)$: constructor ground term where constructor is nested n times

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$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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γ_{Nats}

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- Generator equations** \mathcal{G} for type τ with constructors c, d :

1. Speculating Conjectures

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if $c : \tau_1 \times \dots \times \tau_k \rightarrow \tau$ and $d : \rho_1 \times \dots \times \tau \times \dots \times \rho_b \rightarrow \tau$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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γ_{Nats}

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- Compute most general **typing** for the TRS

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- Generator equations \mathcal{G}** for type τ with constructors c, d :

if $c : \tau_1 \times \dots \times \tau_k \rightarrow \tau$ and $d : \rho_1 \times \dots \times \tau \times \dots \times \rho_b \rightarrow \tau$

then \mathcal{G} contains

$$\gamma_\tau(0) = c(\gamma_{\tau_1}(0), \dots, \gamma_{\tau_k}(0))$$

$$\gamma_\tau(n+1) = d(\gamma_{\rho_1}(0), \dots, \gamma_\tau(n), \dots, \gamma_{\rho_b}(0))$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n+1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

γ_{List}

- Compute most general **typing** for the TRS

$\text{zero} : \text{Nats}, \quad \text{succ} : \text{Nats} \rightarrow \text{Nats}, \quad \text{nil} : \text{List}, \quad \text{cons} : \text{Nats} \times \text{List} \rightarrow \text{List}, \dots$

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$$\gamma_\tau(0) = c(\gamma_{\tau_1}(0), \dots, \gamma_{\tau_k}(0))$$

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1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n+1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n+1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- Compute most general **typing** for the TRS

$\text{zero} : \text{Nats}, \quad \text{succ} : \text{Nats} \rightarrow \text{Nats}, \quad \text{nil} : \text{List}, \quad \text{cons} : \text{Nats} \times \text{List} \rightarrow \text{List}, \dots$

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then \mathcal{G} contains

$$\gamma_\tau(0) = c(\gamma_{\tau_1}(0), \dots, \gamma_{\tau_k}(0))$$

$$\gamma_\tau(n+1) = d(\gamma_{\rho_1}(0), \dots, \gamma_\tau(n), \dots, \gamma_{\rho_b}(0))$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

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- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$

1. Speculating Conjectures

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- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic,

1. Speculating Conjectures

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$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

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- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\text{zero}, \text{cons}^n(\text{zero}, \text{nil})) \xrightarrow{3n+1} \text{nil}$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

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$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0)$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

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$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

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$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0) , \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

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- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$

$$\text{qs}(\gamma_{\text{List}}(n))$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

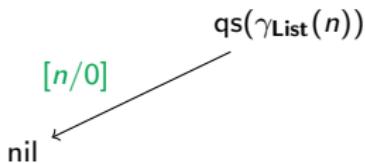
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

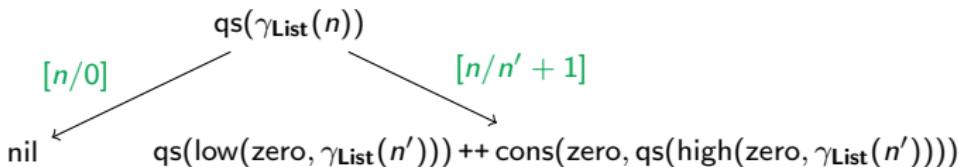
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

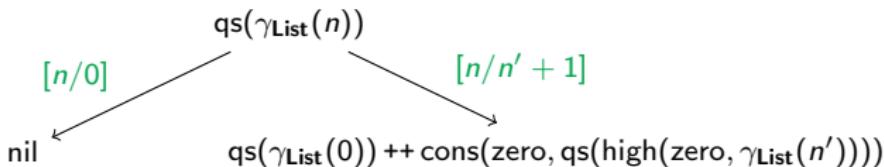
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

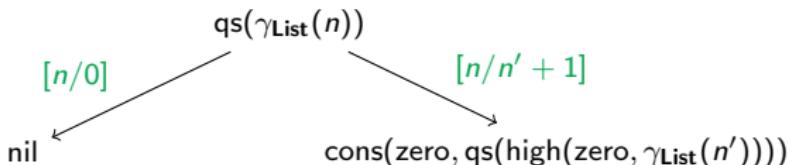
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

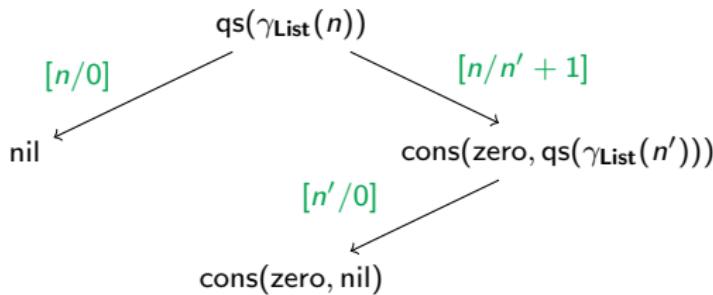
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

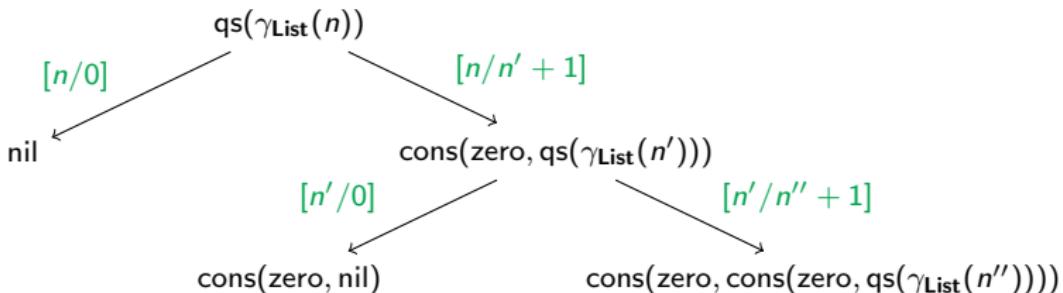
$$\gamma_{\text{Nats}}(n + 1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

$$\gamma_{\text{List}}(n + 1) = \text{cons}(\text{zero}, \gamma_{\text{List}}(n))$$

- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

$$\gamma_{\text{Nats}}(0) = \text{zero}$$

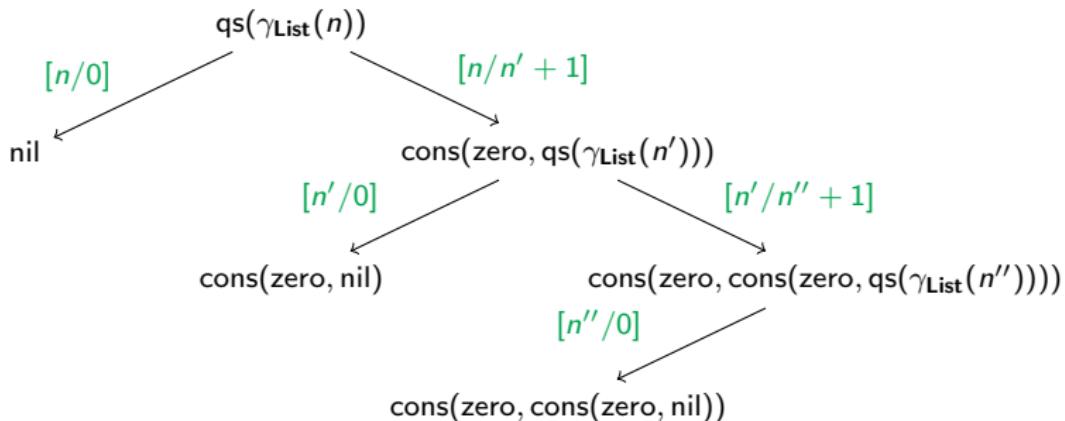
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$$\gamma_{\text{List}}(0) = \text{nil}$$

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- For $f : \tau_1 \times \dots \times \tau_k \rightarrow \tau$, speculate a conjecture by **narrowing** $f(\gamma_{\tau_1}(n_1), \dots, \gamma_{\tau_k}(n_k))$ using rewrite rules, generator equations, integer arithmetic, already proven rewrite lemmas:

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

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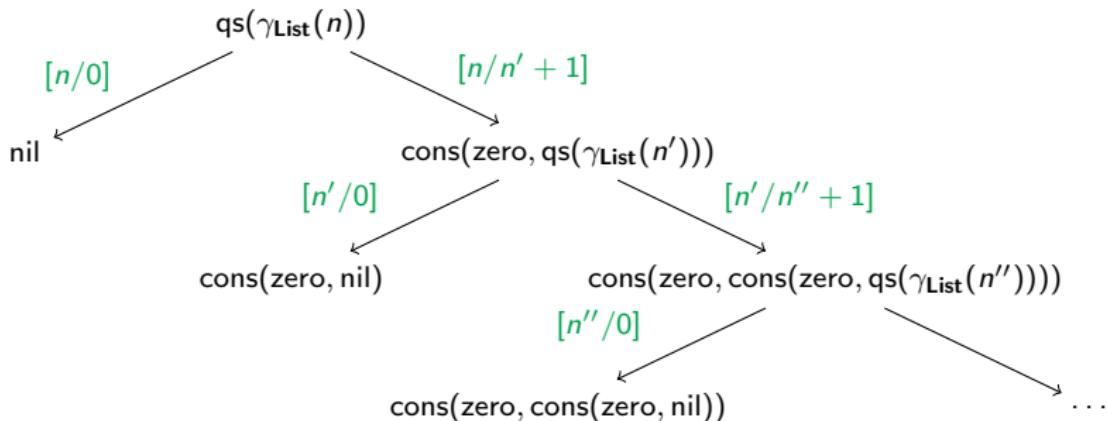
$$\gamma_{\text{Nats}}(n+1) = \text{succ}(\gamma_{\text{Nats}}(n))$$

$$\gamma_{\text{List}}(0) = \text{nil}$$

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$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0), \quad \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

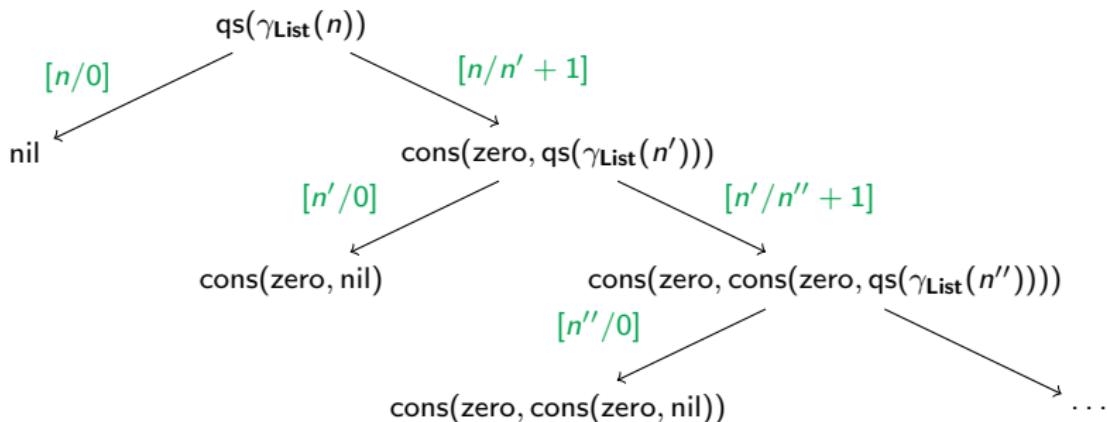
$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

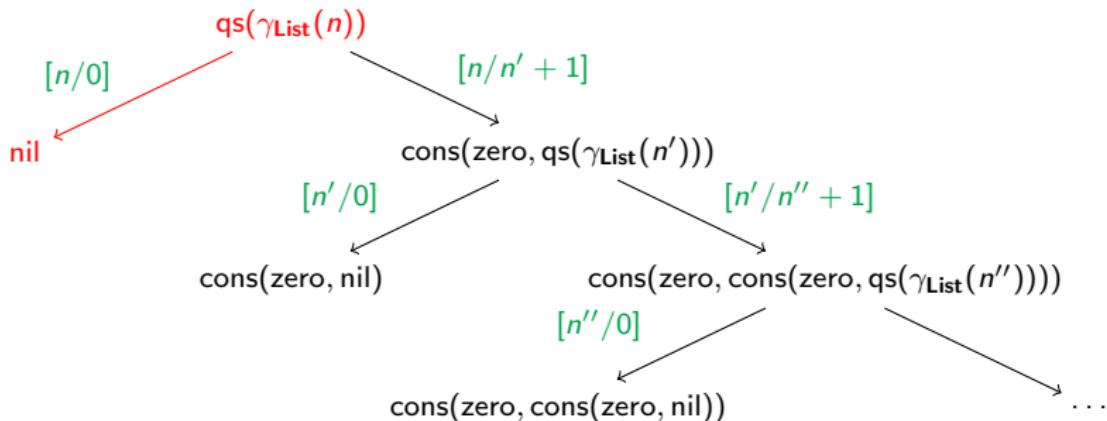
$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

$$\text{qs}(\gamma_{\text{List}}(n))[n/0] \rightarrow^* \gamma_{\text{List}}(0)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

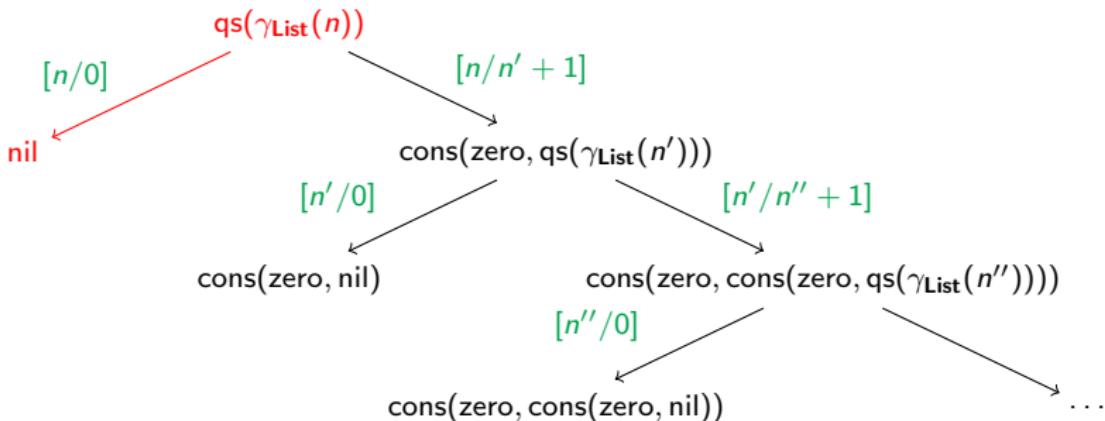
$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d : number of recursive qs-rule applications

$$d = 0 \quad \text{qs}(\gamma_{\text{List}}(n))[n/0] \rightarrow^* \gamma_{\text{List}}(0)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

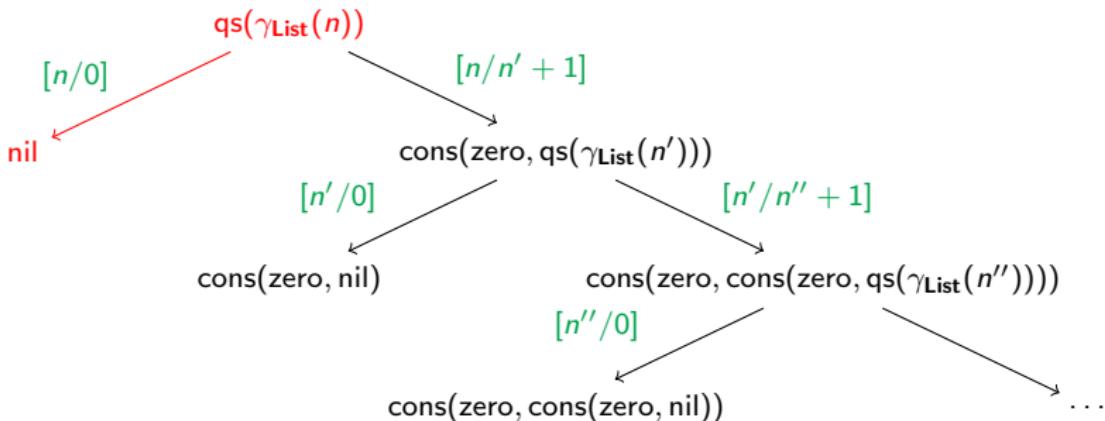
$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d : number of recursive qs-rule applications

$$d = 0$$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$



1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

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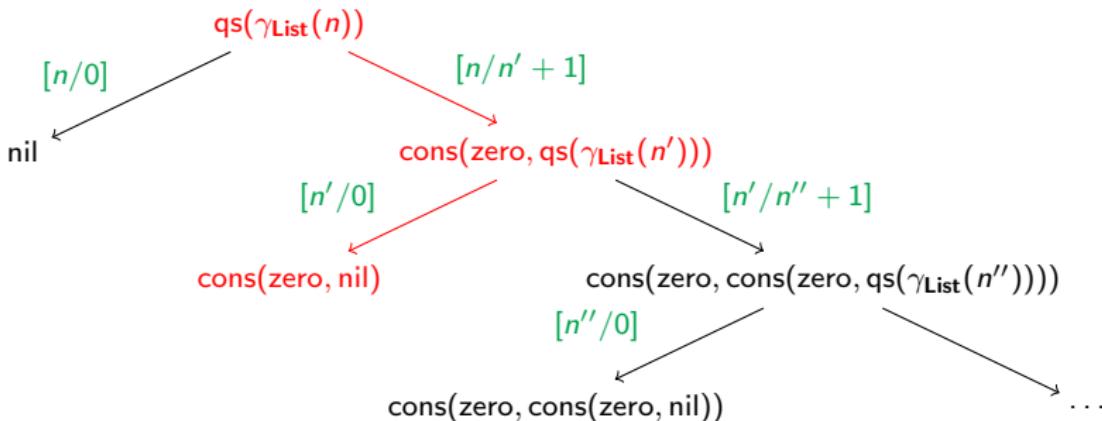
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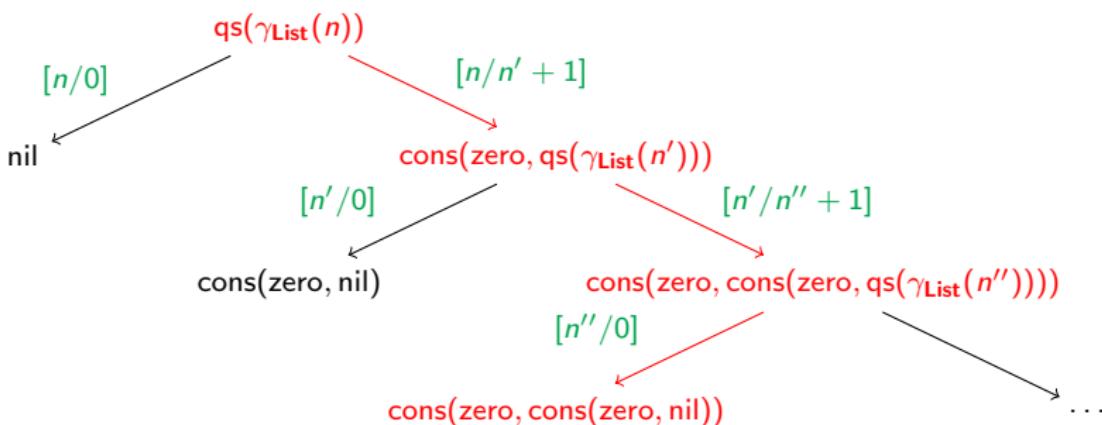
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- Speculate conjecture**

1. Speculating Conjectures

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$$d = 2 \quad \text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

- Speculate conjecture**

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$
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Recursion depth d: number of recursive qs-rule applications

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$$d = 2 \quad \text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(\text{pol}_{\text{left}}(d))) \rightarrow^* \gamma_{\text{List}}(\text{pol}_{\text{right}}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d: number of recursive qs-rule applications

$$d = 0$$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{left}(d) = 0$$

$$pol_{right}(d) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(pol_{left}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{right}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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Recursion depth d: number of recursive qs-rule applications

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$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{left}(0) = 0$$

$$pol_{right}(0) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(pol_{left}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{right}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d: number of recursive qs-rule applications

$$d = 0$$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{left}(0) = 0$$

$$pol_{right}(0) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$pol_{left}(1) = 1$$

$$pol_{right}(1) = 1$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(pol_{left}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{right}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d: number of recursive qs-rule applications

$$d = 0$$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{\text{left}}(0) = 0$$

$$pol_{\text{right}}(0) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$pol_{\text{left}}(1) = 1$$

$$pol_{\text{right}}(1) = 1$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

$$pol_{\text{left}}(2) = 2$$

$$pol_{\text{right}}(2) = 2$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(pol_{\text{left}}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{\text{right}}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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- Every path from root to leaf yields a **sample conjecture**

Recursion depth d: number of recursive qs-rule applications

$$d = 0$$

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$$pol_{left}(0) = 0$$

$$pol_{right}(0) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$pol_{left}(1) = 1$$

$$pol_{right}(1) = 1$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

$$pol_{left}(2) = 2$$

$$pol_{right}(2) = 2$$

- Speculate conjecture**

$$\text{qs}(\gamma_{\text{List}}(pol_{left}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{right}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*
- e constraints can be solved by polynomial of degree $e - 1$

1. Speculating Conjectures

$$qs(nil) \rightarrow nil$$

$$low(x, nil) \rightarrow nil$$

$$if(tt, x, cons(y, ys)) \rightarrow low(x, ys)$$

$$qs(cons(x, xs)) \rightarrow qs(low(x, xs)) ++ cons(x, qs(high(x, xs)))$$

$$low(x, cons(y, ys)) \rightarrow if(x \leq y, x, cons(y, ys))$$

$$if(ff, x, cons(y, ys)) \rightarrow cons(y, low(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d: number of recursive qs-rule applications

$$d = 0$$

$$qs(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{left}(0) = 0$$

$$pol_{right}(0) = 0$$

$$d = 1$$

$$qs(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$pol_{left}(1) = 1$$

$$pol_{right}(1) = 1$$

$$d = 2$$

$$qs(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

$$pol_{left}(2) = 2$$

$$pol_{right}(2) = 2$$

- Speculate conjecture

$$qs(\gamma_{\text{List}}(pol_{left}(d))) \rightarrow^* \gamma_{\text{List}}(pol_{right}(d))$$

- replace *numbers* in sample conjectures by polynomial with variable *d*
- e constraints can be solved by polynomial of degree $e - 1$

$$pol_{left}(d) = d$$

$$pol_{right}(d) = d$$

1. Speculating Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

- Every path from root to leaf yields a **sample conjecture**

Recursion depth d : number of recursive qs-rule applications

$$d = 0$$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow^* \gamma_{\text{List}}(0)$$

$$pol_{\text{left}}(0) = 0$$

$$pol_{\text{right}}(0) = 0$$

$$d = 1$$

$$\text{qs}(\gamma_{\text{List}}(1)) \rightarrow^* \gamma_{\text{List}}(1)$$

$$pol_{\text{left}}(1) = 1$$

$$pol_{\text{right}}(1) = 1$$

$$d = 2$$

$$\text{qs}(\gamma_{\text{List}}(2)) \rightarrow^* \gamma_{\text{List}}(2)$$

$$pol_{\text{left}}(2) = 2$$

$$pol_{\text{right}}(2) = 2$$

- Speculate conjecture

$$\text{qs}(\gamma_{\text{List}}(\boxed{} \ d \ \boxed{})) \rightarrow^* \gamma_{\text{List}}(\boxed{} \ d \ \boxed{})$$

- replace *numbers* in sample conjectures by polynomial with variable d

- e constraints can be solved by polynomial of degree $e - 1$

$$pol_{\text{left}}(d) = \boxed{d}$$

$$pol_{\text{right}}(d) = \boxed{d}$$

Runtime Complexity Analysis of TRSs

$$\begin{array}{ll} \text{qs(nil)} \rightarrow \text{nil} & \text{qs(cons}(x, xs)\text{)} \rightarrow \text{qs}(\text{low}(x, xs)) \text{++ cons}(x, \text{qs}(\text{high}(x, xs))) \\ \text{low}(x, \text{nil}) \rightarrow \text{nil} & \text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys)) \\ \text{if(tt, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{low}(x, ys) & \text{if(ff, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{cons}(y, \text{low}(x, ys)) \end{array}$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

$$\text{qs}(\quad \gamma_{\text{List}}(d) \quad) \rightarrow^* \gamma_{\text{List}}(d)$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

- relation between length of rewrite sequence and size of first term in sequence

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

⑤ Improvements

⑥ Implementation and Experiments

Runtime Complexity Analysis of TRSs

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Runtime Complexity Analysis of TRSs

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

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$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

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$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

① Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments

2. Proving Conjectures

$\text{qs}(\text{nil}) \rightarrow \text{nil}$

$\text{low}(x, \text{nil}) \rightarrow \text{nil}$

$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$

$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$

$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$

$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$

Conjecture speculated

$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \quad \rightarrow^* \quad \gamma_{\text{List}}(n)$

- **rewriting**

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \quad \rightarrow^* \quad \gamma_{\text{List}}(n)$$

• rewriting

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

$$\begin{array}{lcl} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(n) \end{array}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \quad \rightarrow^* \quad \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

$$\begin{array}{lcl} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(n) \end{array}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

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Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\begin{array}{lcl} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) & \xrightarrow{3n+1} & \gamma_{\text{List}}(n) \end{array}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0))$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \equiv_{\mathcal{G}} \text{qs}(\text{nil})$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

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$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \equiv_{\mathcal{G}} \text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\begin{array}{l} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n) \end{array}$$

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$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \equiv_{\mathcal{G}} \text{qs}(\text{nil}) \rightarrow \text{nil} \equiv_{\mathcal{G}} \gamma_{\text{List}}(0)$$

$$\begin{array}{l} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{array}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

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$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1))$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\begin{aligned} \text{qs}(\gamma_{\text{List}}(n+1)) &\rightarrow \\ \text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) &+ + \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \end{aligned}$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

• rewriting

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

• induction on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))))$$

$$\begin{aligned} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) &\rightarrow^{3n+1} \gamma_{\text{List}}(n) \end{aligned}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n)))$$

$$\begin{array}{l} \text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0) \\ \text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n) \end{array}$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0)$$

$$\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(n)$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n)))$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^* \gamma_{\text{List}}(n)$$

- **rewriting**

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

- **induction on n**

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

$$\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(n)$$

2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Conjecture speculated

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^* \gamma_{\text{List}}(n)$$

• rewriting

with rewrite rules, already proven rewrite lemmas, generator equations, integer arithmetic

• induction on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n + 1) \rightarrow$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0)$$

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2. Proving Conjectures

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

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Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

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Conjecture proved

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

① Speculate Conjectures

$$\text{qs}(\quad \gamma_{\text{List}}(n) \quad) \rightarrow^* \gamma_{\text{List}}(n)$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments

Runtime Complexity Analysis of TRSs

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$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

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Rewrite Lemma

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$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

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$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n + 1)$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

$$\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(n)$$

3. Inferring Bounds for Rewrite Lemmas

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^{\pi(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 1 + 3n + 1 + 3n + 1 + 1 + 1$$

- induction** on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n + 1)$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

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3. Inferring Bounds for Rewrite Lemmas

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$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

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Rewrite Lemma

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^{\pi(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

- induction** on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n + 1)$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

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3. Inferring Bounds for Rewrite Lemmas

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Rewrite Lemma

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^{\pi(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

- induction** on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n + 1)$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

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3. Inferring Bounds for Rewrite Lemmas

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$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

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$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\gamma_{\text{List}}(n)) \rightarrow^{\pi(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- induction** on n

Induction Base: $n = 0$

$$\text{qs}(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$\text{qs}(\gamma_{\text{List}}(n + 1)) \rightarrow$$

$$\text{qs}(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\dots))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$\text{qs}(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \text{qs}(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(0)$$

$$\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow^{3n+1} \gamma_{\text{List}}(n)$$

$$\gamma_{\text{List}}(n + 1) \rightarrow$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

Rewrite Lemma

$$qs(\gamma_{\text{List}}(n)) \rightarrow^{rt(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

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$$ih = 1$$

- **induction** on n

Induction Base: $n = 0$

$$qs(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$qs(\gamma_{\text{List}}(n+1)) \rightarrow$$

$$qs(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\text{high}(\dots))) \rightarrow$$

$$qs(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

$$qs(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\gamma_{\text{List}}(n))) \rightarrow$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \rightarrow$$

$$\gamma_{\text{List}}(n+1)$$

$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0)$$

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3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

$$\begin{aligned} rt(0) &= 1 \\ rt(n+1) &= rt(n) + 6n + 5 \end{aligned}$$

Rewrite Lemma

$$qs(\gamma_{\text{List}}(n)) \rightarrow^{rt(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- **induction** on n

Induction Base: $n = 0$

$$qs(\gamma_{\text{List}}(0)) \rightarrow \gamma_{\text{List}}(0)$$

Induction Step: $n > 0$

$$qs(\gamma_{\text{List}}(n+1)) \rightarrow$$

$$qs(\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n))) ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\text{high}(\dots))) \rightarrow$$

$$qs(\gamma_{\text{List}}(0)) ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\text{high}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)))) \rightarrow$$

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$$\text{nil} ++ \text{cons}(\gamma_{\text{Nats}}(0), qs(\gamma_{\text{List}}(n))) \rightarrow_{\text{IH}}$$

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$$\text{low}(\gamma_{\text{Nats}}(0), \gamma_{\text{List}}(n)) \xrightarrow{3n+1} \gamma_{\text{List}}(0)$$

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3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

$$\begin{aligned} rt(0) &= 1 \\ rt(n+1) &= rt(n) + 6n + 5 \end{aligned}$$

Rewrite Lemma

$$qs(\gamma_{\text{List}}(n)) \rightarrow^{rt(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + ih^{n-1} \cdot is(0) + ih^{n-2} \cdot is(1) + \dots + ih \cdot is(n-2) + is(n-1)$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

$$\begin{aligned} rt(0) &= 1 \\ rt(n+1) &= rt(n) + 6n + 5 \end{aligned}$$

Rewrite Lemma

$$qs(\gamma_{\text{List}}(n)) \rightarrow^{rt(n)} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

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$$is(n) = 6n + 5$$

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$$ih = 1$$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 1 + \sum_{i=0}^{n-1} (6i + 5)$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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- **Asymptotic Runtime** of Rewrite Lemmas

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

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- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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$$ib = 1$$

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$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) = is(n - 1)$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

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$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$

- if $ih = 1$: $rt(n) = ib + \sum_{i=0}^{n-1} is(i)$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

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- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

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$$is(n) = 6n + 5$$

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$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$

- if $ih = 1$: $rt(n) = ib + \sum_{i=0}^{n-1} (t_0 + t_1 \cdot i + t_2 \cdot i^2 + \dots + t_{\deg(is)} \cdot i^{\deg(is)})$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

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$$qs(\gamma_{\text{List}}(n)) \rightarrow^{3n^2+2n+1} \gamma_{\text{List}}(n)$$

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$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$

- if $ih = 1$: $rt(n) = ib + t_0 \cdot \sum_{i=0}^{n-1} i^0 + t_1 \cdot \sum_{i=0}^{n-1} i^1 + \dots + t_{\deg(is)} \cdot \sum_{i=0}^{n-1} i^{\deg(is)}$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

- **Recurrence equations** for rt :

$$\begin{aligned} rt(0) &= ib \\ rt(n+1) &= ih \cdot rt(n) + is(n) \end{aligned}$$

$$\begin{aligned} rt(0) &= 1 \\ rt(n+1) &= rt(n) + 6n + 5 \end{aligned}$$

Rewrite Lemma

$$qs(\gamma_{\text{List}}(n)) \rightarrow^{3n^2+2n+1} \gamma_{\text{List}}(n)$$

- Induction proof yields **runtime** $rt(n)$

ib : length of induction base

$$ib = 1$$

$is(n)$: length of induction step without IH-applications

$$is(n) = 6n + 5$$

ih : number of IH-applications in induction step

$$ih = 1$$

- **Asymptotic Runtime** of Rewrite Lemmas

- if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$ degree 1

- if $ih = 1$: $rt(n) = ib + t_0 \cdot \overbrace{\sum_{i=0}^{n-1} i^0} + t_1 \cdot \sum_{i=0}^{n-1} i^1 + \dots + t_{\deg(is)} \cdot \sum_{i=0}^{n-1} i^{\deg(is)}$

- **Explicit Runtime** of Rewrite Lemmas

$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1$$

3. Inferring Bounds for Rewrite Lemmas

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- if $ih > 1$: $rt(n) \geq \sum_{i=0}^{n-1} ih^{n-1-i}$

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$$rt(n) = ih^n \cdot ib + \sum_{i=0}^{n-1} ih^{n-1-i} \cdot is(i)$$

$$rt(n) = 3n^2 + 2n + 1 \in \Omega(n^2)$$

Runtime Complexity Analysis of TRSs

$$\begin{array}{ll} \text{qs(nil)} \rightarrow \text{nil} & \text{qs(cons}(x, xs)\text{)} \rightarrow \text{qs}(\text{low}(x, xs)) \text{++ cons}(x, \text{qs}(\text{high}(x, xs))) \\ \text{low}(x, \text{nil}) \rightarrow \text{nil} & \text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys)) \\ \text{if(tt, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{low}(x, ys) & \text{if(ff, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{cons}(y, \text{low}(x, ys)) \end{array}$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- ① Speculate Conjectures
- ② Prove Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

- ③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

- ④ Infer Bounds for TRSs

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

- ⑤ Improvements

- ⑥ Implementation and Experiments

Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

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Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2 + 2n + 1} \text{cons}^n(\text{zero}, \text{nil})$$

4. Inferring Bounds for TRSs

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Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

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Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2 + 2n + 1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

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Rewrite Lemma

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- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$
 - $\text{rc}_{\mathcal{R}}$ defined w.r.t. *size* of start term

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. *size* of start term
- $\text{size } sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

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- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

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$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size $\text{sz}(n)$

with runtime $\text{rt}(n)$

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$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size $\text{sz}(n) = 2n+2$ with runtime $\text{rt}(n) = 3n^2 + 2n + 1$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size $\text{sz}(n) = 2n+2$ with runtime $\text{rt}(n) = 3n^2 + 2n + 1$

- inverse function $\text{sz}^{-1}(n)$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size $\text{sz}(\text{sz}^{-1}(n))$ with runtime $\text{rt}(\text{sz}^{-1}(n))$

- inverse function $\text{sz}^{-1}(n)$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size n with runtime $\text{rt}(\text{sz}^{-1}(n))$

- inverse function $\text{sz}^{-1}(n)$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size n with runtime $\text{rt}(\text{sz}^{-1}(n))$

- inverse function $\text{sz}^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size

n

with runtime $\text{rt}(\text{sz}^{-1}(n)) = \text{rt}\left(\frac{n-2}{2}\right)$

- inverse function $\text{sz}^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size n with runtime $\text{rt}(\text{sz}^{-1}(n)) = \frac{3}{4}n^2 - 2n + 2$

- inverse function $\text{sz}^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

$$\text{qs}(\text{nil}) \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $\text{rc}_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Explicit Lower Bound** for $\text{rc}_{\mathcal{R}}$

$$\text{rc}_{\mathcal{R}}(n) \geq \text{rt}(\text{sz}^{-1}(n))$$

- $\text{rc}_{\mathcal{R}}$ defined w.r.t. size of start term

- size $\text{sz}(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute

$$|\text{qs}(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$$

There is a term of size n with runtime $\text{rt}(\text{sz}^{-1}(n)) = \frac{3}{4}n^2 - 2n + 2$

- inverse function $\text{sz}^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$

$i\text{b}$: length of induction base
 $is(n)$: length of induction step
 $i\text{h}$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Explicit Lower Bound** for $rc_{\mathcal{R}}$ $rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$
- $rc_{\mathcal{R}}$ defined w.r.t. size of start term
- size $sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute $|qs(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$

There is a term of size n with runtime $rt(sz^{-1}(n)) = \frac{3}{4}n^2 - 2n + 2$

- inverse function $sz^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$

$i\text{b}$: length of induction base
 $is(n)$: length of induction step
 $i\text{h}$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$ $rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$
- $rc_{\mathcal{R}}$ defined w.r.t. size of start term
- size $sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$ easy to compute $|qs(\text{cons}^n(\text{zero}, \text{nil}))| = 2n+2$

There is a term of size n with runtime $rt(sz^{-1}(n)) = \frac{3}{4}n^2 - 2n + 2$

- inverse function $sz^{-1}(n) = \frac{n-2}{2}$

4. Inferring Bounds for TRSs

Asymptotic Runtime:

| | |
|--|--------------------------------------|
| if $i\hbar = 0$: $rt(n) \in \Omega(n^{\deg(is)})$ | $i\hbar$: length of induction base |
| if $i\hbar = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$ | $is(n)$: length of induction step |
| if $i\hbar > 1$: $rt(n) \in \Omega(i\hbar^n)$ | $i\hbar$: number of IH-applications |

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

4. Inferring Bounds for TRSs

Asymptotic Runtime:

| | | |
|---|---------------|---------------------------|
| if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$ | $i\text{b}$: | length of induction base |
| if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$ | $is(n)$: | length of induction step |
| if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$ | $i\text{h}$: | number of IH-applications |

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))| \quad \deg(sz) = \text{max degree of polynomials } s_1, \dots, s_k$$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $ih = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $ih = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $ih > 1$: $rt(n) \in \Omega(ih^n)$

ib : length of induction base
 $is(n)$: length of induction step
 ih : number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

$\deg(sz) = \max$ degree of polynomials s_1, \dots, s_k

$$sz(n) = |qs(\gamma_{\text{List}}(n))|$$

4. Inferring Bounds for TRSs

Asymptotic Runtime:

| | | |
|---|---------------|---------------------------|
| if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$ | $i\text{b}$: | length of induction base |
| if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$ | $is(n)$: | length of induction step |
| if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$ | $i\text{h}$: | number of IH-applications |

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

$\deg(sz) = \max$ degree of polynomials s_1, \dots, s_k

$$sz(n) = |qs(\gamma_{\text{List}}(n))|$$

$\deg(sz) = \deg(n) = 1$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\hbar = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\hbar = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\hbar > 1$: $rt(n) \in \Omega(i\hbar^n)$

$i\hbar$: length of induction base
 $is(n)$: length of induction step
 $i\hbar$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

$\deg(sz)$ = max degree of polynomials s_1, \dots, s_k

$$sz(n) = |qs(\gamma_{\text{List}}(n))|$$

$\deg(sz) = \deg(n) = 1$

if $i\hbar = 0$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

if $i\hbar = 1$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

if $i\hbar > 1$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

4. Inferring Bounds for TRSs

Asymptotic Runtime:

- if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
- if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
- if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$

$i\text{h}$: length of induction base
 $is(n)$: length of induction step
 $i\text{h}$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

$\deg(sz)$ = max degree of polynomials s_1, \dots, s_k

$$sz(n) = |qs(\gamma_{\text{List}}(n))|$$

$\deg(sz) = \deg(n) = 1$

if $i\text{h} = 0$: $rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)}{\deg(sz)}})$

if $i\text{h} = 1$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

if $i\text{h} > 1$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\hbar = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\hbar = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\hbar > 1$: $rt(n) \in \Omega(i\hbar^n)$

$i\hbar$: length of induction base
 $is(n)$: length of induction step
 $i\hbar$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$

- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))|$$

$\deg(sz)$ = max degree of polynomials s_1, \dots, s_k

$$sz(n) = |qs(\gamma_{\text{List}}(n))|$$

$\deg(sz) = \deg(n) = 1$

if $i\hbar = 0$: $rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)}{\deg(sz)}})$

if $i\hbar = 1$: $rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)+1}{\deg(sz)}})$

if $i\hbar > 1$: $rc_{\mathcal{R}}(n) \in \Omega(rt(sz^{-1}(n)))$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$

$i\text{h}$: length of induction base
 $is(n)$: length of induction step
 $i\text{h}$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$\begin{array}{ll} sz(n) = |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))| & \deg(sz) = \max \text{ degree of polynomials } s_1, \dots, s_k \\ sz(n) = |qs(\gamma_{\text{List}}(n))| & \deg(sz) = \deg(n) = 1 \end{array}$$

$$\text{if } i\text{h} = 0: \quad rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)}{\deg(sz)}})$$

$$\text{if } i\text{h} = 1: \quad rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)+1}{\deg(sz)}})$$

$$\text{if } i\text{h} > 1: \quad rc_{\mathcal{R}}(n) \in \Omega(i\text{h}^{\sqrt[\deg(sz)]{n}})$$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\text{h} = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\text{h} = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\text{h} > 1$: $rt(n) \in \Omega(i\text{h}^n)$

$i\text{h}$: length of induction base
 $is(n)$: length of induction step
 $i\text{h}$: number of IH-applications

Rewrite Lemma

$$qs(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- $rc_{\mathcal{R}}(n)$: length of longest \rightarrow -sequence with basic term t where $|t| \leq n$
- **Asymptotic Lower Bound** for $rc_{\mathcal{R}}$

$$rc_{\mathcal{R}}(n) \geq rt(sz^{-1}(n))$$

$$\begin{aligned} sz(n) &= |f(\gamma_{\tau_1}(s_1), \dots, \gamma_{\tau_k}(s_k))| & \deg(sz) &= \max \text{ degree of polynomials } s_1, \dots, s_k \\ sz(n) &= |qs(\gamma_{\text{List}}(n))| & \deg(sz) &= \deg(n) = 1, \quad \deg(is) = 1 \end{aligned}$$

$$\text{if } i\text{h} = 0: \quad rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)}{\deg(sz)}})$$

$$\text{if } i\text{h} = 1: \quad rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{\deg(is)+1}{\deg(sz)}})$$

$$\text{if } i\text{h} > 1: \quad rc_{\mathcal{R}}(n) \in \Omega(i\text{h}^{\sqrt[\deg(sz)]{n}})$$

4. Inferring Bounds for TRSs

Asymptotic Runtime: if $i\hbar = 0$: $rt(n) \in \Omega(n^{\deg(is)})$
if $i\hbar = 1$: $rt(n) \in \Omega(n^{\deg(is)+1})$
if $i\hbar > 1$: $rt(n) \in \Omega(i\hbar^n)$

$i\hbar$: length of induction base
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if $i\hbar > 1$: $rc_{\mathcal{R}}(n) \in \Omega(i\hbar^{\sqrt[\deg(sz)]{n}})$

$$rc_{\mathcal{R}}(n) \in \Omega(n^{\frac{1}{\frac{1}{\deg(sz)} + 1}})$$

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$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments

Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

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5. Improvements

$\mathcal{R} :$ $\text{intlist}(\text{zero}) \rightarrow \text{nil}$ $\text{intlist}(\text{succ}(x)) \rightarrow \text{cons}(x, \text{intlist}(x))$

5. Improvements

$$\mathcal{R} : \quad \text{intlist(zero)} \rightarrow \text{nil} \qquad \qquad \text{intlist(succ}(x)\text{)} \rightarrow \text{cons}(x, \text{intlist}(x))$$

- $\text{intlist}(\text{succ}^n(\text{zero})) \xrightarrow{n+1} \text{cons}(\text{succ}^{n-1}(\text{zero}), \dots, \text{cons}(\text{succ}(\text{zero}), \text{cons}(\text{zero}, \text{nil})))$

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- rewrite lemmas can only represent homogeneous data objects

5. Improvements: Argument Filtering

$$\mathcal{R} : \quad \text{intlist(zero)} \rightarrow \text{nil} \qquad \text{intlist(succ}(x)\text{)} \rightarrow \text{cons}(x, \text{intlist}(x))$$

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5. Improvements: Argument Filtering

$$\mathcal{R}' : \quad \text{intlist(zero)} \rightarrow \text{nil} \quad \text{intlist(succ}(x)\text{)} \rightarrow \text{cons}(\text{ } \text{ intlist}(x))$$

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- $$\mathcal{R} : f(\text{cons}(tt, xs)) \rightarrow f(\text{cons}(ff, xs))$$

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$$\mathcal{R} : f(\text{cons}(tt, xs)) \rightarrow f(\text{cons}(ff, xs)) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1)$$

5. Improvements: Argument Filtering

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 - TRS is left-linear

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$$\mathcal{R}' : f(xs, xs) \rightarrow f(\text{cons}(\text{xs}), \text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

- filtering does not create free variables on rhs

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$$\mathcal{R}' : \text{intlist(zero)} \rightarrow \text{nil} \quad \text{intlist(succ}(x)) \rightarrow \text{cons}(\text{intlist}(x))$$

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 - TRS is left-linear
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$$\mathcal{R}' : f(\text{cons}(\text{xs})) \rightarrow f(\text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

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$$\mathcal{R} : f(\text{cons}(x, xs)) \rightarrow f(xs)$$

5. Improvements: Argument Filtering

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$$\mathcal{R}' : f(\text{cons}(\text{xs})) \rightarrow f(\text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

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$$\mathcal{R}' : f(\text{cons}(\text{xs})) \rightarrow f(\text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

- TRS is left-linear

$$\mathcal{R}' : f(xs, xs) \rightarrow f(\text{cons}(\text{xs}), \text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

- filtering does not create free variables on rhs

$$\mathcal{R}' : f(\text{cons}(x)) \rightarrow f(xs) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(n)$$

5. Improvements: Argument Filtering

$$\mathcal{R}' : \text{intlist(zero)} \rightarrow \text{nil} \quad \text{intlist(succ}(x)) \rightarrow \text{cons}(\text{intlist}(x))$$

- **Argument Filtering**

Rewrite Lemma $\text{intlist}(\text{succ}^n(\text{zero})) \rightarrow^{n+1} \text{cons}^n(\text{nil})$

Lower Bound $\text{rc}_{\mathcal{R}'}(n) \in \Omega(n)$

- rewrite lemmas can only represent homogeneous data objects

- **Solution:** filter away the first argument of cons

- Argument Filtering is **sound** if

- filtering does not remove functions on lhs

$$\mathcal{R}' : f(\text{cons}(\text{xs})) \rightarrow f(\text{cons}(\text{xs})) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(1) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

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- filtering does not create free variables on rhs

$$\mathcal{R}' : f(\text{cons}(x)) \rightarrow f(xs) \quad \text{rc}_{\mathcal{R}}(n) \in \Omega(n) \quad \text{rc}_{\mathcal{R}'}(n) \in \Omega(\omega)$$

5. Improvements

$$\mathcal{R} : \quad f(succ(x)) \rightarrow succ(f(x))$$

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5. Improvements: Indefinite Rewrite Lemmas

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- **Indefinite Rewrite Lemmas**

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- **Indefinite Rewrite Lemmas**

- $f(succ^n(\text{zero})) \rightarrow^n succ^n(f(\text{zero}))$
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- **Solution:** allow *indefinite rewrite lemmas*

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Rewrite Lemma $f(succ^n(\text{zero})) \rightarrow^n *$

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Rewrite Lemma $f(succ^n(\text{zero})) \rightarrow^n *$

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- rewrite lemmas cannot represent n constructors above defined symbol
- **Solution:** allow *indefinite rewrite lemmas*
- adaption of techniques for speculating and proving conjectures

Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^* \text{cons}^n(\text{zero}, \text{nil})$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \rightarrow^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

$$rc_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments

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⑤ Improvements

⑥ Implementation and Experiments

Innermost Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

$$\text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys))$$

$$\text{if}(\text{ff}, x, \text{cons}(y, ys)) \rightarrow \text{cons}(y, \text{low}(x, ys))$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \xrightarrow{i}^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \xrightarrow{i}^* \text{cons}^n(\text{zero}, \text{nil})$$

② Prove Conjectures

③ Infer Bounds for Rewrite Lemmas

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④ Infer Bounds for TRSs

$$\text{irc}_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments

Innermost Runtime Complexity Analysis of TRSs

$$\begin{array}{ll} \text{qs(nil)} \rightarrow \text{nil} & \text{qs(cons}(x, xs)\text{)} \rightarrow \text{qs}(\text{low}(x, xs)) \text{++ cons}(x, \text{qs}(\text{high}(x, xs))) \\ \text{low}(x, \text{nil}) \rightarrow \text{nil} & \text{low}(x, \text{cons}(y, ys)) \rightarrow \text{if}(x \leq y, x, \text{cons}(y, ys)) \\ \text{if(tt, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{low}(x, ys) & \text{if(ff, } x, \text{cons}(y, ys)\text{)} \rightarrow \text{cons}(y, \text{low}(x, ys)) \end{array}$$

Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \xrightarrow{i}^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

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⑤ Improvements

⑥ Implementation and Experiments

6. Implementation and Experiments

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- Implementation in **AProVE**

<http://aprove.informatik.rwth-aachen.de>

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6. Implementation and Experiments

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| $\text{irc}_{\mathcal{R}}(n)$ | $\Omega(1)$ | $\Omega(n)$ | $\Omega(n^2)$ | $\Omega(n^3)$ | $\Omega(n^{>3})$ | $\Omega(2^n)$ | $\Omega(3^n)$ | $\Omega(\omega)$ |
|-------------------------------|-------------|-------------|---------------|---------------|------------------|---------------|---------------|------------------|
| $\mathcal{O}(1)$ | (51) | — | — | — | — | — | — | — |
| $\mathcal{O}(n)$ | 65 | 201 | — | — | — | — | — | — |
| $\mathcal{O}(n^2)$ | 5 | 57 | 17 | — | — | — | — | — |
| $\mathcal{O}(n^3)$ | — | 10 | 3 | 8 | — | — | — | — |
| $\mathcal{O}(n^{>3})$ | 3 | 3 | 1 | — | — | — | — | — |
| $\mathcal{O}(2^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(3^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(\omega)$ | 78 | 293 | 47 | 6 | — | 10 | 1 | (87) |

6. Implementation and Experiments

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- first tool for **lower** complexity bounds of (innermost) runtime complexity
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- Experiments on 808 TRSs from the TPDB

| $\text{irc}_{\mathcal{R}}(n)$ | $\Omega(1)$ | $\Omega(n)$ | $\Omega(n^2)$ | $\Omega(n^3)$ | $\Omega(n^{>3})$ | $\Omega(2^n)$ | $\Omega(3^n)$ | $\Omega(\omega)$ |
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| $\mathcal{O}(1)$ | (51) | — | — | — | — | — | — | — |
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| $\mathcal{O}(2^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(3^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(\omega)$ | 78 | 293 | 47 | 6 | — | 10 | 1 | (87) |

- lower bounds** inferred for 657 TRSs (81 %)

6. Implementation and Experiments

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- Experiments on 808 TRSs from the TPDB

| $\text{irc}_{\mathcal{R}}(n)$ | $\Omega(1)$ | $\Omega(n)$ | $\Omega(n^2)$ | $\Omega(n^3)$ | $\Omega(n^{>3})$ | $\Omega(2^n)$ | $\Omega(3^n)$ | $\Omega(\omega)$ |
|-------------------------------|-------------|-------------|---------------|---------------|------------------|---------------|---------------|------------------|
| $\mathcal{O}(1)$ | (51) | — | — | — | — | — | — | — |
| $\mathcal{O}(n)$ | 65 | 201 | — | — | — | — | — | — |
| $\mathcal{O}(n^2)$ | 5 | 57 | 17 | — | — | — | — | — |
| $\mathcal{O}(n^3)$ | — | 10 | 3 | 8 | — | — | — | — |
| $\mathcal{O}(n^{>3})$ | 3 | 3 | 1 | — | — | — | — | — |
| $\mathcal{O}(2^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(3^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(\omega)$ | 78 | 293 | 47 | 6 | — | 10 | 1 | (87) |

- lower bounds** inferred for 657 TRSs (81 %)
- upper bounds** inferred for 373 TRSs (46 %)

6. Implementation and Experiments

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| $\text{irc}_{\mathcal{R}}(n)$ | $\Omega(1)$ | $\Omega(n)$ | $\Omega(n^2)$ | $\Omega(n^3)$ | $\Omega(n^{>3})$ | $\Omega(2^n)$ | $\Omega(3^n)$ | $\Omega(\omega)$ |
|-------------------------------|-------------|-------------|---------------|---------------|------------------|---------------|---------------|------------------|
| $\mathcal{O}(1)$ | (51) | — | — | — | — | — | — | — |
| $\mathcal{O}(n)$ | 65 | 201 | — | — | — | — | — | — |
| $\mathcal{O}(n^2)$ | 5 | 57 | 17 | — | — | — | — | — |
| $\mathcal{O}(n^3)$ | — | 10 | 3 | 8 | — | — | — | — |
| $\mathcal{O}(n^{>3})$ | 3 | 3 | 1 | — | — | — | — | — |
| $\mathcal{O}(2^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(3^n)$ | — | — | — | — | — | — | — | — |
| $\mathcal{O}(\omega)$ | 78 | 293 | 47 | 6 | — | 10 | 1 | (87) |

- lower bounds** inferred for 657 TRSs (81 %)
- upper bounds** inferred for 373 TRSs (46 %)
- tight bounds** inferred for 226 TRSs (28 %)

Innermost Runtime Complexity Analysis of TRSs

$$\text{qs(nil)} \rightarrow \text{nil}$$

$$\text{low}(x, \text{nil}) \rightarrow \text{nil}$$

$$\text{if}(\text{tt}, x, \text{cons}(y, ys)) \rightarrow \text{low}(x, ys)$$

$$\text{qs}(\text{cons}(x, xs)) \rightarrow \text{qs}(\text{low}(x, xs)) ++ \text{cons}(x, \text{qs}(\text{high}(x, xs)))$$

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Rewrite Lemma

$$\text{qs}(\text{cons}^n(\text{zero}, \text{nil})) \xrightarrow{i}^{3n^2+2n+1} \text{cons}^n(\text{zero}, \text{nil})$$

① Speculate Conjectures

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- *rewrite lemma* describes family of rewrite sequences

④ Infer Bounds for TRSs

$$\text{irc}_{\mathcal{R}}(n) \in \Omega(n^2)$$

- relation between length of rewrite sequence and size of first term in sequence

⑤ Improvements

⑥ Implementation and Experiments