Exercise 1 (Simple Prolog):  
(1.5 + 2 + 1.5 = 5 points)

Consider the following Prolog program, which realizes simple terms built from variables and the function symbol plus.

variable(x).
variable(y).
term(X) :- variable(X).
term(plus(X,Y)) :- term(X), term(Y).

a) Implement a predicate uses(TERM, A) in Prolog which is true iff TERM is a term built with plus and A is an argument of the outermost plus symbol of TERM, i.e., term(TERM) is true and moreover, TERM = plus(A,_) or TERM = plus(_,A).

b) Implement a predicate contains(TERM, X) in Prolog which is true iff TERM is a term built with plus and X is a subterm of TERM. In other words, contains(TERM, X) is true if uses(TERM, X) is true or if there are \( n > 0 \) elements \( Y_1, \ldots, Y_n \) such that the following statements are true:

- uses(TERM, \( Y_1 \))
- uses(\( Y_n \), X)
- uses(\( Y_{i-1} \), \( Y_i \)) for all \( i \in \{2, \ldots, n\} \).

Make sure that the evaluation of all queries \(?- contains(t1, t2)\) terminates if \( t1 \) and \( t2 \) are ground terms.

c) List the first five answers that Prolog computes for the following queries, in the order that Prolog gives them. Try to solve this part of the exercise without the help of a computer.

1. \(?- term(plus(x, X)).\)
2. \(?- uses(plus(x, y), X).\)
3. \(?- contains(plus(plus(x, x), x), X).\)
Exercise 2 (Syntax):  
(2 + 1 = 3 points)

Consider the following Prolog program.

```prolog
robot(wall_e).
robot(otto).
robot(c3po).
robot(r2d2).
robot(android(looks_like_a_human)).
robot(android(looks_like_a_machine)).
can_walk(c3po).
can_drive(r2d2).
can_drive(wall_e).
same_story(c3po,r2d2).
same_story(wall_e,otto).

can_move(X) :- can_walk(X).
can_move(X) :- can_drive(X).
same_story(X,Y):- robot(X), robot(Y), same_story(Y,X).
```

a) Construct the corresponding sets of formulas, predicate symbols, function symbols, and variables based on the program.

b) Give Prolog queries corresponding to the following questions:
   - “Which robots can both walk and drive?”
   - “Which pairs of robots can move, and are both part of the same story?”

Exercise 3 (Induction):  
(4 points)

Let $t$ be an arbitrary term. Then the depth $d(t)$ of $t$ is defined as follows: $d(X) = 1$ if $X$ is a variable. Otherwise we have for $n \geq 0$ that $d(f(t_1, \ldots, t_n)) = 1 + \max\{d(t_1), \ldots, d(t_n)\}$.

Show by structural induction that for every term $t$ and every variable renaming $\sigma$ we have $d(t) = d(\sigma(t))$.

Exercise 4 (Semantics):  
(3 + 3 + 3 = 9 points)

Let $(\Sigma, \Delta)$ be a signature with $\Sigma = \Sigma_0, \Delta = \Delta_1 \cup \Delta_3, \Delta_1 = \{\text{even}\}, \Delta_3 = \{\text{plus}\}$ and $\Sigma_0 = \{0, 1, 2\}$.

Moreover, let

- $\Phi = \{\text{even}(0), \text{even}(2), \forall X, Y, Z \text{ even}(Z) \land \text{plus}(X, Y, Z) \rightarrow \text{even}(X) \land \text{even}(Y)\}$,
- $\varphi = \forall Z \text{ plus}(2, 1, Z) \land \text{even}(Z) \rightarrow \text{even}(1)$,
- $S = (\mathbb{N}, \alpha)$ with
  - $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 2$,
  - $\alpha_{\text{plus}} = \{(x, y, z) \in \mathbb{N}^3 \mid x + y = z\}$,
  - $\alpha_{\text{even}} = \{2 \cdot i \mid i \in \mathbb{N}\}$.

Prove or disprove the following statements.

You may use that addition on natural numbers is commutative.

a) $S \models \varphi$

b) $\models \varphi$

c) $\Phi \models \varphi$