Exercise 1 (Resolution): (2 points)

Consider again the following logic program from Exercise Sheet 2.

\[
\begin{align*}
\text{smaller}(0, s(X)). \\
\text{smaller}(s(X), s(Y)) & : - \text{smaller}(X, Y).
\end{align*}
\]

and the query

\[
? - \text{smaller}(s(0), s(s(s(0)))).
\]

Show that the formulas \( \varphi_1 \) and \( \varphi_2 \) corresponding to the logic program entail the formula \( \varphi \) corresponding to the query (i.e., \( \{ \varphi_1, \varphi_2 \} \models \varphi \)) using the resolution algorithm in predicate logic.

Solution:

Following the solution from Exercise Sheet 2 we know that the corresponding formula in Skolem normal form is

\[
\forall X, Y, Z \; \psi
\]

with

\[
\psi = \text{smaller}(0, s(X)) \land (\neg \text{smaller}(Y, Z) \lor \text{smaller}(s(Y), s(Z))) \land \neg \text{smaller}(s(0), s(s(s(0))))
\]

Now we transform this formula into CNF and obtain the clauses:

\[
\begin{align*}
K_1 &= \{ \text{smaller}(0, s(X)) \} \\
K_2 &= \{ \neg \text{smaller}(Y, Z), \text{smaller}(s(Y), s(Z)) \} \\
K_3 &= \{ \neg \text{smaller}(s(0), s(s(s(0)))) \}
\end{align*}
\]

By deriving the empty clause as shown below we prove that \( \{ \varphi_1, \varphi_2 \} \models \varphi \) holds.

\[
\begin{align*}
&\{ \text{smaller}(0, s(X)) \} \\
&\{ \neg \text{smaller}(Y, Z), \text{smaller}(s(Y), s(Z)) \} \\
&\{ \neg \text{smaller}(s(0), s(s(s(0)))) \}
\end{align*}
\]

\[
\begin{align*}
&\{ X/s(0) \} \\
&\{ \neg \text{smaller}(0, s(s(0))) \}
\end{align*}
\]

Exercise 2 (Lifting Lemma): (3 points)

Consider the clauses \( \{ \text{smaller}(0, s(X)) \}, \{ \neg \text{smaller}(Y, Z), \text{smaller}(s(Y), s(Z)) \} \) (based on Exercise 1). These clauses can be resolved to \( R := \{ \text{smaller}(s(0), s(s(X))) \} \) as follows:

For this resolution step, find all ground instances \( A', B', \) and \( R' \) of \( A, B, \) and \( R \) (using substitution with ground terms built from the function symbols \( s \) and \( 0 \)), such that we have

\[
\begin{align*}
&\text{smaller}(s(0), s(s(X))) \\
&\text{smaller}(0, s(X)) \\
&\text{smaller}(s(X), s(Y))
\end{align*}
\]
and by the lifting lemma (Lemma 3.4.8) we get:

If there is an infinite number of such ground instances for $A$, $B$, and $R$, give a suitable finite description of these ground instances.

**Solution:**

There is an infinite number of such ground instances $A'$, $B'$, and $R'$. For each $n \in \mathbb{N}_0$ by applying the substitutions

\[
\sigma_A := \{X/s^n(0)\}
\]
\[
\sigma_B := \{Y/0, Z/s^n(0)\}
\]
\[
\sigma_R := \{X/s^n(0)\}
\]

as follows

\[
A' = \sigma_A(A) = \{\text{smaller}(0, s^n(0))\}
\]
\[
B' = \sigma_B(B) = \{\neg\text{smaller}(0, s^n(0)), \text{smaller}(0, s(s^n(0)))\}
\]
\[
R' = \sigma_R(R) = \{\text{smaller}(s(0), s(s^n(0)))\}
\]

we have

Using the lifting lemma we then get

**Exercise 3 (Restrictions of Resolution):** \((2 + 3 + 3 + 2 + 2 + 2 = 14\text{ points})\)

Consider the sets of clauses

\[
\mathcal{K}_1 = \{p(a, f(f(X))), \{p(Y, Z), \neg p(Y, f(f(Z))))\}, \{-p(X, b), q(X)\}, \{-p(X, b), \neg q(X)\}
\]

and

\[
\mathcal{K}_2 = \{p(a, f(f(X))), \{p(X, Y), \neg p(Y, X), \neg p(Y, f(f(Y))))\}, \{-p(f(Z), a)\}
\]

with $a, b \in \Sigma_0$, $f \in \Sigma_1$, $q \in \Delta_1$, and $p \in \Delta_2$. 
a) Derive the empty clause from $\mathcal{K}_1$ using full but not linear resolution (i.e., there must be at least one non-linear resolution step). For each step denote the substitutions used.

b) Derive the empty clause from $\mathcal{K}_1$ using linear but not input resolution. For each step denote the substitutions used.

c) Derive the empty clause from $\mathcal{K}_1$ using input resolution. For each step denote the substitutions used.

d) Derive the empty clause from $\mathcal{K}_2$ using SLD resolution but not binary SLD resolution. For each step denote the substitutions used. In addition, also give the answer substitution.

e) Derive the empty clause from $\mathcal{K}_2$ using binary SLD resolution. For each step denote the substitutions used. In addition, also give the answer substitution.

f) Express $\mathcal{K}_1$ and $\mathcal{K}_2$ as queries, facts, and rules of a logic program.

Solution:

a) 

\[
\begin{align*}
\sigma_1 &= \{p(a,f(f(X)))\} \\
\sigma_2 &= \{p(Y,Z), \neg p(Y,f(f(Z)))\} \\
\sigma_3 &= \{\neg p(X,b), q(X)\} \\
\sigma_4 &= \{\neg p(X,b), \neg q(X)\} \\
\nu_1 &= \{X/X_1\} \\
\nu_2 &= \{X/X_2\}
\end{align*}
\]

$\sigma_1 = \{Y/a, Z/X\}$

$\sigma_2 = \{X_2/X_1\}$

$\sigma_3 = \{X/b, X_1/a\}$

b) 

\[
\begin{align*}
\sigma_1 &= \{p(a,f(f(X)))\} \\
\sigma_2 &= \{p(Y,Z), \neg p(Y,f(f(Z)))\} \\
\nu_1 &= \{X/X_1\} \\
\nu_2 &= \{X/X_2\}
\end{align*}
\]

$\sigma_3 = \{q(X)\}$

$\sigma_4 = \{\neg p(a,b)\}$

$\sigma_1 = \{Y/a, Z/X\}$

$\nu_1 = \{X/X_1\}$

$\sigma_2 = \{X_1/b, X_2/a\}$

$\sigma_3 = \{X/a\}$

$\sigma_4 = \{X/b\}$
c) \[
\{p(a,f(f(X))))\} \quad \{p(Y,Z),\neg p(Y,f(f(Z))))\} \quad \{\neg p(X,b),q(X)\} \quad \{\neg p(X,b),\neg q(X)\}
\]
\[
\sigma_1 = \{X/X_1\} \quad \nu_1 = \{X/X_1\} \quad \sigma_2 \quad \nu_2 = \{X/X_2\} \quad \sigma_3 \quad \{\neg p(X_1,f(f(b))))\}
\]
\[
\sigma_1 = \{X_2/X_1\} \quad \sigma_2 = \{Y/X_1, Z/b\} \quad \sigma_3 = \{X_1/a, X/b\}
\]

d) \[
\{p(a,f(f(X))))\} \quad \{p(X,Y),\neg p(Y,X),\neg p(Y,f(f(Y))))\} \quad \{\neg p(f(Z),a)\}
\]
\[
\sigma_1 \quad \sigma_1 \quad \sigma_2 \quad \sigma_2 \quad \sigma_3 \quad \{\neg p(a,f(f(Z))),\neg p(a,f(f(a))))\}
\]
\[
\sigma_1 = \{X/f(Z), Y/a\} \quad \sigma_2 = \{X/a, Z/f(a)\} \quad \text{Answer substitution: } \{Z/f(a)\}
\]

e) \[
\{p(a,f(f(X))))\} \quad \{p(X,Y),\neg p(Y,X),\neg p(Y,f(f(Y))))\} \quad \{\neg p(f(Z),a)\}
\]
\[
\sigma_1 \quad \sigma_1 \quad \sigma_2 \quad \sigma_2 \quad \sigma_3 \quad \{\neg p(a,f(f(Z)))\}
\]
\[
\sigma_1 = \{X/f(Z), Y/a\} \quad \sigma_2 = \{X/a\} \quad \sigma_3 = \{Z/f(X)\} \quad \text{Answer substitution: } \{Z/f(X)\}
\]
\[
\text{Alternatively: }
\]
Exercise 4 (Multi-Resolution): (2 + 2 = 4 points)

In this exercise we consider an extension of resolution in propositional logic, which we call multi-resolution. Let $K_1$ and $K_2$ be clauses without variables. Then a clause $R$ is a multi-resolvent of $K_1$ and $K_2$ iff for some $n > 0$ there are literals $L_1, \ldots, L_n$ such that $K_1 = K'_1 \uplus \{L_1, \ldots, L_n\}$, $K_2 = K'_2 \uplus \{\overline{L_1}, \ldots, \overline{L_n}\}$, and $R = K'_1 \cup K'_2$. Here, $\uplus$ denotes disjoint union. Thus, $K \uplus K'$ stands for the set $K \cup K'$ and it states that $K \cap K' = \emptyset$. The following diagram illustrates a multi-resolution step:

Please prove or disprove the following statements:

a) Multi-resolution is sound, i.e., there is no satisfiable clause set $K$ without variables from which one can derive $\Box$ by multi-resolution.

b) Multi-resolution is complete, i.e., from any unsatisfiable clause set $K$ without variables one can derive $\Box$ by multi-resolution.

Solution: 

Answer substitution: $\{Z/f(a)\}$

Exercise 4 (Multi-Resolution):

In this exercise we consider an extension of resolution in propositional logic, which we call multi-resolution. Let $K_1$ and $K_2$ be clauses without variables. Then a clause $R$ is a multi-resolvent of $K_1$ and $K_2$ iff for some $n > 0$ there are literals $L_1, \ldots, L_n$ such that $K_1 = K'_1 \uplus \{L_1, \ldots, L_n\}$, $K_2 = K'_2 \uplus \{\overline{L_1}, \ldots, \overline{L_n}\}$, and $R = K'_1 \cup K'_2$. Here, $\uplus$ denotes disjoint union. Thus, $K \uplus K'$ stands for the set $K \cup K'$ and it states that $K \cap K' = \emptyset$. The following diagram illustrates a multi-resolution step:

Please prove or disprove the following statements:

a) Multi-resolution is sound, i.e., there is no satisfiable clause set $K$ without variables from which one can derive $\Box$ by multi-resolution.

b) Multi-resolution is complete, i.e., from any unsatisfiable clause set $K$ without variables one can derive $\Box$ by multi-resolution.

Solution: 

Answer substitution: $\{Z/f(a)\}$
a) Soundness of multi-resolution is refuted by the following counter-example: Let $K_1 = \{p, q\}$ and $K_2 = \{\neg p, \neg q\}$. Then we can resolve $\Box$ in a single multi-resolution step by choosing $L_1 = p$ and $L_2 = q$. However, the clause set $\{K_1, K_2\}$ is satisfiable, as witnessed e.g. by a structure $S$ with $S \models p$ and $S \not\models q$.

b) Completeness of multi-resolution follows from the fact that any propositional resolution step according to Def. 3.3.4 corresponds to a step in multi-resolution with $n = 1$ and from completeness of propositional resolution (Thm. 3.3.7).