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Exercise 1 (Simple Prolog): (1.5 + 2 + 1.5 = 5 points)

Consider the following Prolog program.

```
evolvedFrom(cat,miacis).
evolvedFrom(hyena,miacis).
evolvedFrom(weasel,miacis).
evolvedFrom(cynodictis,miacis).

evolvedFrom(raccoon,cynodictis).
evolvedFrom(bear,cynodictis).
evolvedFrom(tomarctus,cynodictis).

evolvedFrom(fox,tomarctus).
evolvedFrom(wolf,tomarctus).
evolvedFrom(dog,tomarctus).
```

a) Implement a predicate `evolvedFromSameCreature(A,B)` in Prolog which is true if both `A` and `B` evolved from the same creature according to the predicate `evolvedFrom` above. For example, the query `?- evolvedFromSameCreature(fox,wolf)` should yield the answer `true`, whereas `?- evolvedFromSameCreature(cat,dog)` should yield `false`.

b) Implement a predicate `descendsFrom(A,C)` in Prolog which is true if `A` is a descendant of `C`, i.e., `A` either directly evolved from `C` or `A` evolved from a descendant `B` of `C`.

Make sure that the evaluation of all queries `?- descendsFrom(...,...)` terminates.

c) List all answers that Prolog gives for the following queries, in the order that Prolog gives them. Try to solve this part of the exercise without the help of a computer.

1. `?- evolvedFrom(X,tomarctus).`
2. `?- evolvedFromSameCreature(raccoon,X).`
3. `?- descendsFrom(wolf,X).`

Solution:

a)
```
evolvedFromSameCreature(A,B) :- evolvedFrom(A,C), evolvedFrom(B,C).
```

b)
```
descendsFrom(A,C) :- evolvedFrom(A,C).
descendsFrom(A,C) :- evolvedFrom(A,B), descendsFrom(B,C).
```

c)

1.
```
X = fox ;
X = wolf ;
X = dog.
```

2.
```
X = raccoon ;
X = bear ;
X = tomarctus.
```

3.
```
X = tomarctus ;
X = cynodictis ;
X = miacis ;
false.
```

Exercise 2 (Syntax): (2 + 1 = 3 points)

Consider the following Prolog program.

```
eats(rabbit,grass).
eats(grasshopper,grass).
eats(mouse,grass).
eats(mouse,corn).
eats(mouse,grasshopper).
eats(fox,rabbit).
eats(fox,mouse).

plant(grass).
plant(corn).
animal(rabbit).
animal(grasshopper).
animal(mouse).
animal(fox).

has_enemy(X) :- animal(X), eats(Y,X).
competitors(X,Y) :- animal(X), animal(Y), eats(X,Z), eats(Y,Z).
```

a) Construct the corresponding sets of formulas, predicate symbols, function symbols, and variables based on the program.

b) Give Prolog queries corresponding to the following questions:

- "Which plants does the mouse eat?"
- "Which competitors of the grasshopper do eat grasshoppers?"

Solution:

a) $\Phi = \{$

$$\begin{aligned}
&\text{eats(rabbit,grass)},\\
&\text{eats(grasshopper,grass)},\\
&\text{eats(mouse,grass)},\\
&\text{eats(mouse,corn)},\\
&\text{eats(mouse,grasshopper)},\\
&\text{eats(fox,rabbit)},\\
&\text{eats(fox,mouse)},\\
&\text{plant(grass)},\\
&\text{plant(corn)},\\
&\text{animal(rabbit)},\\
&\text{animal(grasshopper)},\\
&\text{animal(mouse)},\\
&\text{animal(fox)},\\
&\forall X, Y \quad \text{animal}(X) \land \text{eats}(Y,X) \to \text{has_enemy}(X),\\
&\forall X, Y, Z \quad \text{animal}(X) \land \text{animal}(Y) \land \text{eats}(X,Z) \land \text{eats}(Y,Z) \to \text{competitors}(X,Y)
\end{aligned}$$

$\}$ over $\Sigma = \Sigma_0 = \{\text{grass}, \text{corn}, \text{rabbit}, \text{grasshopper}, \text{mouse}, \text{fox}\}$, $\Delta_1 = \{\text{plant}, \text{animal}, \text{has\_enemy}\}$, $\Delta_2 = \{\text{eats}, \text{competitors}\}$, $\Delta = \Delta_1 \cup \Delta_2$, and $\mathcal{V} = \{X, Y, Z\}$.

b)
- `?- eats(mouse,X), plant(X).`
- `?- competitors(grasshopper,X), eats(X,grasshopper).`

Exercise 3 (Induction): (3 points)

Let t be an arbitrary term. Then the size $|t|$ of t is defined as follows. $|X| = 1$ if X is a variable. Otherwise we have for $n \geq 0$ that $|f(t_1, \ldots, t_n)| = 1 + \Sigma_{i=1}^{n} |t_i|$.
Show by structural induction that for every term t and every variable renaming σ we have $|t| = |\sigma(t)|$.

Solution:

Let t be an arbitrary term and σ be an arbitrary variable renaming. In the induction base, we consider the case that t is a variable. Then we have $|t| = 1$. According to the definition of variable renamings, we have $\sigma(t) \in \mathcal{V}$ with $|\sigma(t)| = 1$, so $|t| = 1 = |\sigma(t)|$ holds.
In the induction step, we consider the case $t = f(t_1, \ldots, t_n)$ with $n \geq 0$. As indution hypothesis, we can assume that we have $|t_i| = |\sigma(t_i)|$ for all $i \in \{1, \ldots, n\}$. We have $\sigma(t) = f(\sigma(t_1), \ldots, \sigma(t_n))$. We also know that $|t| = 1 + \Sigma_{i=1}^{n} |t_i|$. Using the induction hypothesis, we then have $|t| = 1 + \Sigma_{i=1}^{n} |\sigma(t_i)| = |\sigma(t)|$.

Exercise 4 (Semantics): (3 + 3 + 3 = 9 points)

Let (Σ, Δ) be a signature with $\Sigma = \Sigma_0 = \{2, 6\}, \Delta = \Delta_1 \cup \Delta_3, \Delta_1 = \{\mathsf{even}\},$ and $\Delta_3 = \{\mathsf{plus}\}$.
Moreover, let

- $\Phi = \{\mathsf{even}(2), \forall X, Y, Z \quad \mathsf{even}(X) \wedge \mathsf{even}(Y) \wedge \mathsf{plus}(X, Y, Z) \rightarrow \mathsf{even}(Z)\}$,
- $\varphi = \forall Y \quad \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \rightarrow \mathsf{even}(Y)$,
- $S = (\mathbb{N}, \alpha)$ with
  - $\alpha_2 = 2$, $\alpha_6 = 6$,
  - $\alpha_{\mathsf{plus}} = \{(x, y, z) \in \mathbb{N}^3 \mid x + y = z\}$,
  - $\alpha_{\mathsf{even}} = \{2 * i \mid i \in \mathbb{N}\}$.

Prove or disprove the following statements.
You may use that addition on natural numbers is commutative.

a) $S \models \varphi$

b) $\models \varphi$

c) $\Phi \models \varphi$

Solution:

a) Let $I = (\mathbb{N}, \alpha, \beta)$ for some variable assignment β. Since φ contains no free variables, we have:

$$\begin{aligned}
S \models \varphi \quad &\Leftrightarrow \quad I \models \varphi \\
&\Leftrightarrow \quad I \models \forall Y \quad \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \to \mathsf{even}(Y) \\
&\Leftrightarrow \quad \Big(I[\![Y/y]\!] \models \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \to \mathsf{even}(Y)\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad \Big(I[\![Y/y]\!] \models \mathsf{even}(Y) \text{ if } I[\![Y/y]\!] \models \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6)\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad \Big(I[\![Y/y]\!] \models \mathsf{even}(Y) \text{ if } (I[\![Y/y]\!] \models \mathsf{plus}(2, Y, 6) \text{ and } I[\![Y/y]\!] \models \mathsf{even}(6))\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad \Big(y \in \alpha_{\mathsf{even}} \text{ if } ((\alpha_2, y, \alpha_6) \in \alpha_{\mathsf{plus}} \text{ and } \alpha_6 \in \alpha_{\mathsf{even}})\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad \Big(y \in \{2 * i \mid i \in \mathbb{N}\} \text{ if} \\
&\qquad ((2, y, 6) \in \{(x, y, z) \in \mathbb{N}^3 \mid x + y = z\} \text{ and } 6 \in \{2 * i \mid i \in \mathbb{N}\})\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad \Big(y \in \{2 * i \mid i \in \mathbb{N}\} \text{ if } (y = 4 \text{ and } 6 \in \{2 * i \mid i \in \mathbb{N}\})\Big) \text{ for all } y \in \mathbb{N} \\
&\Leftrightarrow \quad y \in \{2 * i \mid i \in \mathbb{N}\} \text{ if } y = 4
\end{aligned}$$

This implication is obviously true since 4 is divisible by two.

b) We choose the following interpretation $I' = (\{2, 3, 6\}, \alpha', \beta)$ with some variable assignment β and $\alpha'_2 = 2, \alpha'_6 = 6, \alpha'_{\mathsf{plus}} = \{(2, 3, 6)\}, \alpha'_{\mathsf{even}} = \{2, 6\}$. Then we have:

$$\begin{aligned}
I' \models \varphi \quad &\Leftrightarrow \quad I' \models \forall Y \quad \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \to \mathsf{even}(Y) \\
&\Leftrightarrow \quad \Big(I'[\![Y/y]\!] \models \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \to \mathsf{even}(Y)\Big) \text{ for all } y \in \{2, 3, 6\} \\
&\Rightarrow \quad I'[\![Y/3]\!] \models \mathsf{plus}(2, Y, 6) \wedge \mathsf{even}(6) \to \mathsf{even}(Y) \\
&\Leftrightarrow \quad \text{if } I'[\![Y/3]\!] \models \mathsf{plus}(2, Y, 6) \text{ and } I'[\![Y/3]\!] \models \mathsf{even}(6), \text{ then } I'[\![Y/3]\!] \models \mathsf{even}(Y) \\
&\Leftrightarrow \quad \text{if } (\alpha'_2, 3, \alpha'_6) \in \alpha'_{\mathsf{plus}} \text{ and } \alpha'_6 \in \alpha'_{\mathsf{even}}, \text{ then } 3 \in \alpha'_{\mathsf{even}} \\
&\Leftrightarrow \quad \text{if } (2, 3, 6) \in \{(2, 3, 6)\} \text{ and } 6 \in \{2, 6\}, \text{ then } 3 \in \{2, 6\}
\end{aligned}$$

This is obviously wrong. Thus, the statement is disproved as I' is not a model of φ.

c) We choose the interpretation I' from the last exercise part again. Now we have:

$$\begin{aligned}
I' \models \mathsf{even}(2) \quad &\Leftrightarrow \quad \alpha'_2 \in \alpha'_{\mathsf{even}} \\
&\Leftrightarrow \quad 2 \in \{2, 6\}
\end{aligned}$$

This is obviously true.

Furthermore, we have:

$$\begin{aligned}
&I' \models \forall X, Y, Z \quad \mathsf{even}(X) \wedge \mathsf{even}(Y) \wedge \mathsf{plus}(X, Y, Z) \to \mathsf{even}(Z) \\
\Leftrightarrow \quad &\Big(I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(X) \wedge \mathsf{even}(Y) \wedge \mathsf{plus}(X, Y, Z) \to \mathsf{even}(Z)\Big) \text{ for all } x, y, z \in \{2, 3, 6\} \\
\Leftrightarrow \quad &\Big(I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(Z) \text{ if } I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(X) \wedge \mathsf{even}(Y) \wedge \mathsf{plus}(X, Y, Z)\Big) \\
&\quad \text{for all } x, y, z \in \{2, 3, 6\} \\
\Leftrightarrow \quad &\Big(I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(Z) \text{ if } I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(X) \text{ and } I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{even}(Y) \\
&\quad \text{and } I'[\![X/x, Y/y, Z/z]\!] \models \mathsf{plus}(X, Y, Z)\Big) \text{ for all } x, y, z \in \{2, 3, 6\} \\
\Leftrightarrow \quad &\Big(z \in \alpha'_{\mathsf{even}} \text{ if } x \in \alpha'_{\mathsf{even}} \text{ and } y \in \alpha'_{\mathsf{even}} \text{ and } (x, y, z) \in \alpha'_{\mathsf{plus}}\Big) \text{ for all } x, y, z \in \{2, 3, 6\} \\
\Leftrightarrow \quad &\Big(z \in \{2, 6\} \text{ if } x \in \{2, 6\} \text{ and } y \in \{2, 6\} \text{ and } (x, y, z) \in \{(2, 3, 6)\}\Big) \text{ for all } x, y, z \in \{2, 3, 6\}
\end{aligned}$$

There is no $y \in \{2, 6\}$ such that $(x, y, z) \in \{(2, 3, 6)\}$. Therefore, the implication is true.

Hence, we have $I' \models \Phi$. However, from part b) we know that $I' \not\models \varphi$. Thus, the statement is disproved.