# Semantics by lub's of Approximations

```
fact :: Int -> Int
fact  = \x -> if x <= 0 then 1 else fact(x-1) * x
```

Regard non-recursively defined approximations

$$\begin{aligned}
\texttt{fact}_0 &= \texttt{\textbackslash x -> bot} \\
\texttt{fact}_1 &= \texttt{\textbackslash x -> if x <= 0 then 1 else fact}_0(\texttt{x}-1) * \texttt{x} \\
\texttt{fact}_2 &= \texttt{\textbackslash x -> if x <= 0 then 1 else fact}_1(\texttt{x}-1) * \texttt{x} \\
&\ldots
\end{aligned}$$

Thus: $\texttt{fact}_{i+1} = \texttt{ff fact}_i$ where

```
ff :: (Int -> Int) -> (Int -> Int)
ff g  = \x -> if x <= 0 then 1 else g(x-1) * x
```

Define semantics *fact* of `fact` as

$$\mathit{fact} = \sqcup\{\mathit{fact}_0, \mathit{fact}_1, \ldots\} = \sqcup\{\mathit{ff}^{\,i}(\bot) \mid i \in \mathbb{N}\}$$

# Semantics by Least Fixpoints

```
fact :: Int -> Int
fact  = \x -> if x <= 0 then 1 else fact(x-1) * x
```

Semantics $\mathit{fact}$ of `fact` should *satisfy* the defining equation

$$\mathit{fact} = \underbrace{\texttt{\textbackslash x -> if x <= 0 then 1 else } \mathit{fact}(\mathtt{x} - 1) * \mathtt{x}}_{\mathit{ff}\,(\mathit{fact})}$$

where

```
ff :: (Int -> Int) -> (Int -> Int)
ff g  = \x -> if x <= 0 then 1 else g(x-1) * x
```

$\mathit{fact}$ should be undefined unless program enforces its definedness.
Define semantics $\mathit{fact}$ of `fact` as

$$\mathit{fact} = \mathrm{lfp}\ \mathit{ff}$$

# Fixpoint Theorem

Both semantics are equivalent:

$$\mathrm{lfp}\; \mathit{ff} \quad = \quad \sqcup\{\mathit{ff}^{\,i}(\perp) \mid i \in \mathbb{N}\}$$

**Thm. 2.1.17 (*Fixpoint Theorem*)**
Let $\sqsubseteq$ be a cpo on $D$ and let $f : D \to D$ be continuous.
Then $f$ has a least fixpoint and we have $\mathrm{lfp}\, f \;=\; \sqcup\{f^i(\perp)|i \in \mathbb{N}\}$.

**Proof.**
$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq f^3(\perp) \sqsubseteq \ldots$ by monotonicity of $f$

So $\{f^i(\perp)|i \in \mathbb{N}\}$ is a chain
and $\sqcup\{f^i(\perp)|i \in \mathbb{N}\}$ exists by completeness of $\sqsubseteq$.

To show: $\sqcup\{f^i(\perp)|i \in \mathbb{N}\}$ is the least fixpoint of $f$