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Exam Term Rewriting Systems

First Name: ______________________________

Last Name: ______________________________

Immatriculation Number: ______________________________

Course of Studies (please mark exactly one):

∘ **Informatik Bachelor** ∘ **Informatik Master**

∘ **Mathematik Bachelor** ∘ **Mathematik Master** ∘ **Other:** ______________________

	Maximal Points	Achieved Points
Exercise 1	24	
Exercise 2	24	
Exercise 3	10	
Exercise 4	14	
Exercise 5	16	
Exercise 6	16	
Exercise 7	16	
Total	120	
Grade	-	

Notes:

- On every sheet please give your **first name**, **last name**, and **immatriculation number**.
- You must solve the exam **without** consulting any **extra documents** (e.g., course notes).
- Make sure your answers are readable. Do not use **red or green pens or pencils**.
- Please answer the exercises on the **exercise sheets**. If needed, also use the back sides of the exercise sheets.
- Answers on extra sheets can only be accepted if they are clearly marked with your name, your immatriculation number, and the **exercise number**.
- **Cross out** text that should not be considered in the evaluation.
- Students that try to cheat **do not pass** the exam.
- At the end of the exam, please return **all sheets together with the exercise sheets**.
- You can give your answers **in English or German**.

Exercise 1 (Theoretical Foundations): (6 × 4 = 24 points)

Give a short proof sketch or a counterexample for each of the following statements:

a) The question whether $s \equiv_{\mathcal{E}} t$ holds is semi-decidable for every set $\mathcal{E}$ of equations between terms.

b) For every terminating term rewrite system $\mathcal{R}$, $\to_{\mathcal{R}}^{+}$ is a simplification order.

c) For every set M and every relation $\to \subseteq M \times M$ such that every $p \in M$ has at most one $\to$-normal form, we have that $\to$ is confluent.

d) Let Σ be a signature with $\mathsf{c} \in \Sigma_2$, let x, y be variables. If $\{\mathsf{c}(x,y) \equiv x, \mathsf{c}(x,y) \equiv y\} \subseteq \mathcal{E}$, then for all terms s, t we have $s \equiv_{\mathcal{E}} t$.

e) The embedding order on terms is normalizing.

f) A relation $\to$ is called *strongly confluent* iff for all p, s, and t with $p \to s$ and $p \to t$ there is a q with $s \to^{=} q$ and $t \to^{=} q$. Here, $s \to^{=} q$ means that $s \to q$ or $s = q$ holds. Then every confluent relation is also strongly confluent.

Exercise 2 (Equivalence Classes): (8 + 12 + 4 = 24 points)

a) Let $s \sim t$ hold for two terms s and t iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s| = |t|$. Here, $|s|$ is the *size* of the term s where $|x| = 1$ for any variable $x \in \mathcal{V}$ and $|f(s_1, \ldots, s_n)| = 1 + |s_1| + \cdots + |s_n|$ for any function symbol $f \in \Sigma$.

Please show that $\sim$ is an equivalence relation and that all equivalence classes w.r.t. $\sim$ are finite.

Hints:

- Remember that we only consider finite signatures Σ.

b) Please show that the word problem is decidable for each set of equations $\mathcal{E}$ where $\equiv_{\mathcal{E}} \subseteq \sim$. To this end, describe a decision procedure which decides for arbitrary given input terms s and t and a set of equations $\mathcal{E}$ with $\equiv_{\mathcal{E}} \subseteq \sim$ whether $s \equiv_{\mathcal{E}} t$ holds.

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

c) Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{f}, \mathsf{g}, \mathcal{O}\}$.

$$\begin{aligned} \mathsf{f}(x, y) &\equiv \mathsf{f}(y, x) \\ \mathsf{g}(x, \mathcal{O}, y, z) &\equiv \mathsf{f}(x, \mathsf{f}(y, z)) \\ \mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) &\equiv \mathsf{g}(y, \mathcal{O}, \mathcal{O}, x) \end{aligned}$$

Prove or disprove: $\mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) \equiv_{\mathcal{E}} \mathsf{f}(x, \mathsf{f}(\mathcal{O}, y))$.

Hints:

- You may use that $\equiv_{\mathcal{E}} \subseteq \sim$ holds.

Exercise 3 (Diamond Lemma): (10 points)

Let M be some set and $\to \subseteq M \times M$ some well-founded, locally confluent relation.
Prove the diamond lemma, i.e., that $\to$ is confluent.

Exercise 4 (Termination): (11 + 3 = 14 points)

Prove or disprove termination of the following term rewrite systems either by means of a reduction order or by a counterexample. In case you prove termination, explicitly state which order you used (including precedence and status if appropriate) and which checks you have to perform for the termination proof (you may do this e.g. with the proof tree notation from the homework exercises where embedding is considered as one step). Here, x, y, xs, and ys denote variables while all other identifiers denote function symbols. Intuitively, mullength multiplies the lengths of the two argument lists.

a)

$$\begin{aligned}
\mathsf{mullength}(\mathsf{Nil}, ys) &\to \mathcal{O}\\
\mathsf{mullength}(\mathsf{Cons}(x, xs), ys) &\to \mathsf{addlength}(ys, \mathsf{mullength}(ys, xs))\\
\mathsf{addlength}(\mathsf{Nil}, y) &\to y\\
\mathsf{addlength}(\mathsf{Cons}(x, xs), y) &\to \mathsf{addlength}(xs, \mathsf{s}(y))
\end{aligned}$$

b)

$$\begin{aligned}
\mathsf{g}(\mathsf{f}(\mathsf{a}), y) &\to \mathsf{a}\\
\mathsf{g}(x, \mathsf{f}(\mathsf{a})) &\to \mathsf{g}(\mathsf{f}(\mathsf{a}), x)
\end{aligned}$$

Exercise 5 (Confluence): (16 points)

A relation $\to$ is called *semi-confluent* iff for all p, s, and t with $p \to^* s$ and $p \to t$ there is a q with $s \to^* q$ and $t \to^* q$.

Prove that every semi-confluent relation is also confluent.

Hints:

- Use induction (but note that $\to$ is not necessarily well founded).

Exercise 6 (Word Problem): (12 + 4 = 16 points)

Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{plus}, \mathsf{s}, \mathcal{O}\}$.

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \equiv & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \equiv & \mathsf{s}(\mathsf{plus}(x, y)) \\ \mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) & \equiv & \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))) \end{array}$$

Prove or disprove the following equations.

a) $\mathsf{plus}(x, \mathcal{O}) \equiv_{\mathcal{E}} x$

b) $\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) \equiv_{\mathcal{E}} \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), x)$

Exercise 7 (Completion): (16 points)

In this exercise, we consider the signature $\Sigma = \{\mathsf{s}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{b}, \mathsf{c}\}$. Please use the **advanced** completion algorithm from the lecture to generate a convergent TRS that is equivalent to the following set of equations:

$$\{\ \mathsf{f}(x, x, z) \equiv \mathsf{h}(x, \mathsf{s}(x)),\ \ \mathsf{f}(x, y, z) \equiv \mathsf{g}(x, \mathsf{s}(z)),\ \ \mathsf{g}(x, y) \equiv \mathsf{b},\ \ \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b}, y)\ \}$$

As reduction order $\succ$, use the LPO with precedence $\mathsf{s} \sqsupset \mathsf{f} \sqsupset \mathsf{g} \sqsupset \mathsf{h} \sqsupset \mathsf{b} \sqsupset \mathsf{c}$. For each step of the advanced completion algorithm, also indicate which transformation rule you are applying. In this exercise you do not need to give a proof for $\ell \succ r$ if you generate a new rule $\ell \to r$ (but this statement should be true, of course).