Prof. Dr. Jürgen Giesl
Marc Brockschmidt, Carsten Fuhs, Thomas Ströder

Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, May 4th, 2011, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Monotonicity): (1 + 1 + 1 + 1 = 4 points)

Consider the following relations $\sim_1, \ldots, \sim_4 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are monotonic.

a) $s \sim_1 t$ iff the numbers of different variables in s and t are equal, i.e., $|\mathcal{V}(s)| = |\mathcal{V}(t)|$.

b) $s \sim_2 t$ iff s is a subterm of t.

c) $s \sim_3 t$ iff s matches t.

d) $s \sim_4 t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$.

Exercise 2 (Equivalence relations): ((1 + 1 + 1 + 1) + (1 + 1) = 6 points)

a) Consider the following relations $\sim_1, \ldots, \sim_4$. Prove or disprove for each of these relations that they are equivalence relations.

i) $\sim_1 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_1 t \Leftrightarrow |\mathcal{V}(s)| = |\mathcal{V}(t)|$.

ii) $\sim_2 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_2 t \Leftrightarrow \exists \sigma \in SUB(\Sigma, \mathcal{V}) : s\sigma = t\sigma$ (i.e., s and t unify).

iii) Let $m \in \mathbb{N}$ be some fixed natural number. Then $\sim_3^m \subseteq \mathbb{Z} \times \mathbb{Z}$ with $x \sim_3^m y \Leftrightarrow (x - y) \ mod \ m = 0$.

For example, we have $61 \sim_3^{19} 23$, as $(61 - 23) \ mod \ 19 = 38 \ mod \ 19 = 0$.

Hints:

- To save work, you may write $x - y \equiv_m 0$ instead of $(x - y) \ mod \ m = 0$.

iv) Let $\mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\}$ and $(p, q), (u, v) \in \mathbb{Z} \times \mathbb{Z}_{\neq 0}$. Then $\sim_4 \subseteq (\mathbb{Z} \times \mathbb{Z}_{\neq 0}) \times (\mathbb{Z} \times \mathbb{Z}_{\neq 0})$ with $(p, q) \sim_4 (u, v) \Leftrightarrow p \cdot v = u \cdot q$.

For example, we have $(3, 6) \sim_4 (2, 4)$, as $3 \cdot 4 = 12 = 2 \cdot 6$.

b) For each of the relations $\sim_1$ and $\sim_2$, give the smallest equivalence relation (i.e., the transitive-reflexive-symmetric closure) that includes them.

i) Let $M = \{0, 1, 2, 3, 4\}$ and $0 \sim_1 2$, $3 \sim_1 1$, $4 \sim_1 2$ and $2 \sim_1 0$.

ii) Let $\mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\}$. Then $\sim_2 \subseteq \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\neq 0}$ with $x \sim_2 y \Leftrightarrow x + 1 = y$.

Exercise 3 (Equivalence classes): (2 + 4 = 6 points)

a) Let $s \sim t$ hold for two terms s and t iff $\mathcal{V}(s) = \mathcal{V}(t)$ and the number of function symbols in s is the same as the number of function symbols in t.

Please show that $\sim$ is an equivalence relation and that all equivalence classes w.r.t. $\sim$ are finite.

b) Please show that the word problem is decidable for the following set of equations $\mathcal{E}$ over $\Sigma = \Sigma_2 \cup \Sigma_0$ with $\Sigma_2 = \{:, \cup\}$ and $\Sigma_0 = \{a\}$.

$$\begin{aligned} (x : y) \cup z &\equiv x : (y \cup z) \\ x \cup (y \cup z) &\equiv (x \cup y) \cup z \\ x \cup y &\equiv y \cup x \\ x : (y : z) &\equiv y : (x : z) \\ x : (y \cup z) &\equiv y \cup (x : z) \end{aligned}$$

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

Exercise 4 (Syntactic Proofs): (1 + 4 = 5 points)

Consider the following set of equations[1] $\mathcal{E}$:

$$\mathsf{f}(x, \mathsf{f}(y, z)) \equiv \mathsf{f}(\mathsf{f}(x, y), z) \quad (1)$$
$$\mathsf{f}(x, \mathsf{e}) \equiv x \quad (2)$$
$$\mathsf{f}(x, \mathsf{i}(x)) \equiv \mathsf{e} \quad (3)$$
$$\mathsf{f}(\mathsf{i}(x), x) \equiv \mathsf{e} \quad (3)'$$

a) Prove $\mathsf{f}(e, x) \equiv x$ using $\leftrightarrow^*_{\mathcal{E}}$. Mark in each step which part of your term you are replacing and which equation you used for it.

b) Prove $\mathsf{f}(\mathsf{i}(v), \mathsf{i}(u)) \equiv \mathsf{i}(\mathsf{f}(u, v))$ using $\leftrightarrow^*_{\mathcal{E}}$. Mark in each step which part of your term you are replacing and which equation you used for it.

[1] They correspond to the group axioms from the lecture and an additional equation for the operation of inverse elements from the right hand side.