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Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, May 11th, 2011, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!
- This whole exercise sheet is **only** relevant for students attending the **V4 (Diplom Informatik and Diplom Mathematik)** version of the lecture. For all other students, the exercises on this sheet do not contribute to the overall number of points that will be required for the exam qualification.
- Exercises or exercise parts marked with a star are voluntary **challenge** exercises with advanced difficulty. However, they do not contribute to the overall number of points that will be required for the exam qualification or for the Übungsschein, respectively.

Exercise 1 (An Application of Congruence Closure): (2 + 3 + 3 = 8 points)

Consider the following code fragment of an imperative program:

```
a = c;
d = f[f[c]];
f[c] = f[f[f[b]]];
if ( f[b] == a ) {
   (*)
}
```

The fragment has been translated to the following set of term equalities $\mathcal{E}$.

$$\begin{aligned} \mathsf{a} &\equiv \mathsf{c} \\ \mathsf{d} &\equiv \mathsf{f(f(c))} \\ \mathsf{f(c)} &\equiv \mathsf{f(f(f(b)))} \\ \mathsf{f(b)} &\equiv \mathsf{a} \end{aligned}$$

a) Show via $\leftrightarrow_{\mathcal{E}}^*$ that $\mathsf{d} \equiv_{\mathcal{E}} \mathsf{f(c)}$ holds.

b) Show via congruence closure that $\mathsf{d} \equiv_{\mathcal{E}} \mathsf{f(c)}$ holds.

c) Give initial values for the variables `a, b, c, d` and for the array `f`, such that at the position `(*)` the value of `d` is not equal to that of `f[c]`. What is the problem?

Exercise 2 (The Algorithm KONGRUENZABSCHLUSS (CONGRUENCE CLOSURE)): (4 points)

Consider the set of term equalities $\mathcal{E}$ consisting of the following ground identities:

$$\begin{aligned} \mathsf{a} &\equiv \mathsf{b} \\ \mathsf{c} &\equiv \mathsf{f(d)} \\ \mathsf{f(b)} &\equiv \mathsf{g(a)} \\ \mathsf{d} &\equiv \mathsf{c} \\ \mathsf{g(b)} &\equiv \mathsf{d} \end{aligned}$$

Decide $\mathsf{g}(\mathsf{c}) \equiv_{\mathcal{E}} \mathsf{f}(\mathsf{f}(\mathsf{a}))$ using the Algorithm KONGRUENZABSCHLUSS (CONGRUENCE CLOSURE) from the lecture. Give the set S and as intermediate results also the sets L during each iteration of Step 4.

Exercise 3 (Congruence Closure for Validity): (6 + 3* + 3 = 9 + 3* points)

The goal of this exercise is to develop a decision procedure for the validity of (implicitly) universally quantified first-order logic formulas (FO-formulas). FO-formulas consist of term equalities and can be connected by the Boolean operators $\neg, \vee, \wedge$ in the usual way. For example, $\varphi = \neg\big(x \equiv \mathsf{f}(\mathsf{f}(x)) \wedge x \equiv \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))))\big) \vee x \equiv \mathsf{f}(x)$ is a FO-formula with $x \in \mathcal{V}$.
For interpretations $I = (\mathcal{A}, \alpha, \beta)$ the model relationship for FO-formulas is defined in the usual way:

- $I \models \varphi_1 \vee \varphi_2$ iff $I \models \varphi_1$ or $I \models \varphi_2$
- $I \models \varphi_1 \wedge \varphi_2$ iff $I \models \varphi_1$ and $I \models \varphi_2$
- $I \models \neg\varphi$ iff $I \not\models \varphi$
- $I \models u \equiv v$ iff $I(u) = I(v)$

A FO-formula φ is called *valid* iff for all interpretations I we have $I \models \varphi$. A FO-formula φ is called *unsatisfiable* iff no interpretation I with $I \models \varphi$ exists.

a) Use the congruence closure procedure to develop a decision procedure for validity of FO-formulas. Hints:
   - Show how one can reduce validity of FO-formulas with variables to validity of FO-formulas without variables.
   - Reduce validity of FO-formulas to unsatisfiability of several conjunctions of the shape $u_1 \equiv v_1 \wedge \cdots \wedge u_n \equiv v_n \wedge \neg s_1 \equiv t_1 \wedge \cdots \wedge \neg s_m \equiv t_m$.
   - Use the following lemma for ground terms s_i, t_i, u_j, v_j with $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$:[1]
     If there exist algebras $A_1, \ldots, A_m$ with $A_i \models u_1 \equiv v_1 \wedge \cdots \wedge u_n \equiv v_n \wedge \neg s_i \equiv t_i$ $(i \in \{1, \ldots, m\})$, then there exists also an algebra A with $A \models u_1 \equiv v_1 \wedge \cdots \wedge u_n \equiv v_n \wedge \neg s_1 \equiv t_1 \wedge \cdots \wedge \neg s_m \equiv t_m$ (and vice versa).

b) Apply your procedure to show validity of the FO-formula φ given above.

[1] and prove it to obtain the bonus points