Prof. Dr. Jürgen Giesl

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Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, May 25th, 2011, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (3 + 2 + 2 = 7 points)

Consider the following set of equalities $\mathcal{E}$ and the term rewrite system $\mathcal{R}$.

$$\mathcal{E} = \{\, \mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{g}(x)))) \equiv x, \quad x \equiv \mathsf{g}(x), \quad \mathsf{f}(\mathsf{g}(x)) \equiv \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) \,\}$$

$$\mathcal{R} = \{\, \mathsf{f}(\mathsf{f}(x)) \to x, \quad \mathsf{g}(x) \to x \,\}$$

a) Please show that $\mathcal{R}$ is equivalent to $\mathcal{E}$.

b) Please show $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(x)))))))))) \equiv_{\mathcal{E}} \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(x)))))$ only using the relation $\leftrightarrow_{\mathcal{E}}$ and Birkhoff's Theorem (in particular, you must not use $\mathcal{R}$ in this subexercise).

c) Please show $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(x)))))))))) \equiv_{\mathcal{E}} \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(x)))))$ using the algorithm WORTPROBLEM.

Hints:

- $\mathcal{R}$ is convergent.

Exercise 2 (Noetherian Induction): (2 + 4 = 6 points)

Consider the following term rewrite system $\mathcal{R}$, which represents the well-known Ackermann function:

$$\mathsf{ack}(\mathcal{O}, m) \to \mathsf{s}(m) \quad (1)$$

$$\mathsf{ack}(\mathsf{s}(n), \mathcal{O}) \to \mathsf{ack}(n, \mathsf{s}(\mathcal{O})) \quad (2)$$

$$\mathsf{ack}(\mathsf{s}(n), \mathsf{s}(m)) \to \mathsf{ack}(n, \mathsf{ack}(\mathsf{s}(n), m)) \quad (3)$$

a) Choose a relation $\succ \subseteq \{(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \mid n_1, k_1 \in \mathbb{N}\} \times \{(\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \mid n_2, k_2 \in \mathbb{N}\}$ and prove that your $\succ$ is well-founded ("fundiert").

b) Prove that any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^m(\mathcal{O}))$ has the form $\mathsf{s}^\ell(\mathcal{O})$ by noetherian induction using the relation $\succ$ from part a).

Exercise 3 (The Algorithm RIGHT-GROUND TERMINATION): (3 + 2 = 5 points)

Prove or disprove termination of the following term rewrite systems over the signature $\Sigma = \{\mathsf{f}, \mathsf{a}, \mathsf{b}\}$ using the algorithm RIGHT-GROUND TERMINATION from the lecture:

a)

$$\begin{aligned}
\mathsf{f}(\mathsf{f}(x, y), z) &\to \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{b}))\\
\mathsf{f}(\mathsf{a}, \mathsf{f}(x, x)) &\to \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{b}, \mathsf{a}))\\
\mathsf{f}(\mathsf{a}, x) &\to \mathsf{a}\\
\mathsf{f}(x, \mathsf{b}) &\to \mathsf{f}(\mathsf{a}, \mathsf{a})\\
\mathsf{f}(\mathsf{b}, \mathsf{a}) &\to \mathsf{b}
\end{aligned}$$

b)

$$\begin{aligned}
\mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, x)) &\to \mathsf{f}(\mathsf{a}, \mathsf{a})\\
\mathsf{f}(x, \mathsf{f}(\mathsf{a}, \mathsf{f}(x, \mathsf{a}))) &\to \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{b}))))
\end{aligned}$$