Positions $Occ(q)$

- $q|_\varepsilon = q$ and $f(q_1, \ldots, q_i, \ldots, q_n)|_{i\,\pi'} = q_i|_{\pi'}$
- $q[r]_\varepsilon = r$ and $f(q_1, \ldots, q_i, \ldots, q_n)[r]_{i\,\pi'} = f(q_1, \ldots, q_i[r]_{\pi'}, \ldots, q_n)$.

Structural Induction on Positions: Prove $\varphi(\pi)$ for all $\pi \in \mathbb{N}^*$

- Induction Base: $\varphi(\epsilon)$
- Induction Step: $\underbrace{\varphi(\pi')}_{Ind.Hyp.} \;\Rightarrow\; \varphi(i\pi')$ for all $i \in \mathbb{N}, \pi' \in \mathbb{N}^*$

Lemma 3.1.8 ($\equiv_{\mathcal{E}}$ monotonic)

(a) If $A \models s \equiv t$, then $A \models q[s]_\pi \equiv q[t]_\pi$.

(b) If $s \equiv_{\mathcal{E}} t$, then $q[s]_\pi \equiv_{\mathcal{E}} q[t]_\pi$.

Proof of (a):

$$\varphi(\pi) = \begin{array}{l}\text{“For all } A, s, t, q \text{ with } \pi \in Occ(q)\text{, we have:}\\ \text{If } A \models s \equiv t\text{, then } A \models q[s]_\pi \equiv q[t]_\pi\text{.”}\end{array}$$

Induction Base: $\varphi(\varepsilon)$, i.e., $\pi = \varepsilon$

Trivial, since $q[s]_\varepsilon = s$, $q[t]_\varepsilon = t$

Induction Step: $\varphi(i\pi')$, i.e., $\pi = i\pi'$

If $i\pi' \in Occ(q)$, then $q = f(q_1, \ldots, q_i, \ldots, q_n)$.

Thus: $q[s]_\pi = f(q_1, \ldots, q_i[s]_{\pi'}, \ldots, q_n)$

$q[t]_\pi = f(q_1, \ldots, q_i[t]_{\pi'}, \ldots, q_n)$