Prof. Dr. Jürgen Giesl   Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- Please register at `https://aprove.informatik.rwth-aachen.de/trs15/` (http**s**, not http!).
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Friday, October 30th, 2015, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Syntax and Semantics): (2 + 4 = 6 points)

a) Give a set of equalities that describes the functions $\mathsf{ge} : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$ and $\mathsf{odd} : \mathbb{N} \to \mathbb{B}$ with $\mathbb{B} = \{\top, \bot\}$ whose exact semantics should be as follows:

$$\mathsf{ge}(x, y) = \begin{cases} \top & \text{if } x \geq y \\ \bot & \text{otherwise} \end{cases}$$

$$\mathsf{odd}(x) = \begin{cases} \top & \text{if } x \text{ is odd} \\ \bot & \text{otherwise} \end{cases}$$

Thus, $\mathsf{odd}(15) = \top$, $\mathsf{odd}(0) = \bot$, $\mathsf{ge}(5, 10) = \bot$, and $\mathsf{ge}(5, 5) = \top$.

Use the representation of natural numbers presented in the lecture, where 0 is represented by $\mathcal{O} \in \Sigma_0$ and n is represented by applying a successor symbol $\mathsf{s} \in \Sigma_1$ n times (i.e., by $\mathsf{s}^n(\mathcal{O})$). Furthermore, use symbols $\mathsf{true}, \mathsf{false} \in \Sigma_0$ to represent $\top$ resp. $\bot$.

b) Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$, and $\Sigma_2 = \{\mathsf{plus}\}$. Consider $\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$, the set of equations describing addition on our representation of natural numbers.

Prove that $\mathcal{E} \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$.

Hints:

- You can use a model $A = (\mathcal{A}, \alpha)$ where $\mathcal{A}$ does not only consist of $\mathbb{N}$, but also contains additional elements (e.g., all words over some alphabet Π). Then define $\alpha_{\mathsf{plus}}(n, m)$ such that it models addition for $n, m \in \mathbb{N}$, but behaves differently if n or m are not from $\mathbb{N}$ (e.g., such that $\alpha_{\mathsf{plus}}(n, m)$ models concatenation if $n, m \in \Pi^*$).

Exercise 2 (Matching): (2 + 3 = 5 points)

a) Consider the following pairs of terms s and t over the signature $\Sigma = \Sigma_0 \cup \Sigma_2$ with $\Sigma_0 = \{\mathsf{a}\}$ and $\Sigma_2 = \{\mathsf{f}\}$. Moreover, we have $\{x, y, z\} \subseteq \mathcal{V}$ for the set of variables $\mathcal{V}$. If s matches t, then give a suitable matcher σ. Otherwise give a brief (at most two sentences) explanation why there is no matcher.

1. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, \mathsf{a})$
2. $s = \mathsf{f}(y, \mathsf{a})$, $t = \mathsf{f}(\mathsf{a}, x)$
3. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, x)$

4. $s = \mathsf{f}(x, y), t = \mathsf{f}(\mathsf{f}(x, z), \mathsf{f}(x, z))$

b) Let $\sim$ be the matching relation, i.e., for two terms s and t we have $s \sim t$ iff s matches t.

Prove or disprove the following propositions:

1. For all terms s, t, and q we have $s \sim t \wedge t \sim q \implies s \sim q$.
2. For all terms s and t we have $s \neq t \wedge \mathcal{V}(s) = \mathcal{V}(t) \implies s \not\sim t$.

Exercise 3 (Induction): (3 points)

Let $t \in \mathcal{T}(\Sigma, \mathcal{V})$, $\pi \in Occ(t)$, and $\sigma \in SUB(\Sigma, \mathcal{V})$. Show by induction over π that $(t|_\pi)\sigma = (t\sigma)|_\pi$ holds.

Hints:

- In the induction base, prove the proposition for $\pi = \epsilon$.
- In the induction step, consider the case $\pi = i\,\pi'$, where as induction hypothesis, you can assume that $(q|_{\pi'})\mu = (q\mu)|_{\pi'}$ for all $q \in \mathcal{T}(\Sigma, \mathcal{V})$ and all $\mu \in SUB(\Sigma, \mathcal{V})$.

Exercise 4 (Stability): (1 + 1 + 2 = 4 points)

Consider the following relations $\sim_1, \dots, \sim_3 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are stable.

a) $s \sim_1 t$ iff $t \trianglerighteq s$ (i.e., iff s is a subterm of t)

b) $s \sim_2 t$ iff s matches t

c) $s \sim_3 t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$

Hints:

- You can use the lemma proven in Exercise 3.