Prof. Dr. Jürgen Giesl — Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- Please register at `https://aprove.informatik.rwth-aachen.de/trs15/` (http**s**, not http!).
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Friday, November 6th, 2015, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!
- Exercises or exercise parts marked with a star are voluntary **challenge** exercises with advanced difficulty. However, they do not contribute to the overall maximum number of points that you can obtain by this exercise sheet.

Exercise 1 (Monotonicity): (2 + 1 = 3 points)

Consider the following relations $\sim_1, \sim_2, \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are monotonic.

a) $s \sim_1 t$ iff $|s| = |t|$ where for all terms t we have $|t| = 1$ if $t \in \mathcal{V}$ and $|t| = 1 + |t_1| + \cdots + |t_n|$ if $t = f(t_1, \ldots, t_n)$.

b) $s \sim_2 t$ iff s matches t and t matches s.

Exercise 2 (Equivalence relations): (1 + 2 + 1 = 4 points)

a) Prove or disprove: $\sim_1 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_1 t \Leftrightarrow |s| = |t|$ is an equivalence relation.

b) Prove or disprove: $\sim_2 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_2 t$ iff s matches t and t matches s is an equivalence relation.

c) Prove or disprove: For all $\mathcal{E} \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$, $\rightarrow_{\mathcal{E}}^*$ is an equivalence relation.

Exercise 3 (Equivalence classes): (2 + 4* points)

a) Let $s \sim t$ hold for two terms s and t iff $\mathcal{V}(s) = \mathcal{V}(t)$ and the number of function symbols in s is the same as the number of function symbols in t.

Please show that $\sim$ is an equivalence relation and that all equivalence classes w.r.t. $\sim$ are finite. Here, an informal argument is enough.

b) Please show that the word problem is decidable for the following set of equations $\mathcal{E}$ over $\Sigma = \{\mathsf{union}, \mathsf{cons}\}$.

$$\begin{aligned}
\mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \\
\mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) \\
\mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys))
\end{aligned}$$

Note: Since this exercise part was harder than intended, we turned it into a challenge exercise.

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

Exercise 4 (Syntactic Proofs): (2 + 3 = 5 points)

Consider the following set of equations $\mathcal{E}$:

$$\begin{aligned} \mathsf{union}(\mathsf{nil}, xs) &\equiv xs & (1)\\ \mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) & (2)\\ \mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) & (3)\\ \mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) & (4)\\ \mathsf{cons}(x, \mathsf{cons}(x, xs)) &\equiv \mathsf{cons}(x, xs) & (5) \end{aligned}$$

a) Prove $\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil})) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))$ using $\leftrightarrow_{\mathcal{E}}^*$. Mark in each step which part of your term you are replacing and which equation you used for it.

b) Prove $\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, xs)), \mathsf{cons}(z, \mathsf{cons}(y, ys))) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(xs, ys))))$ using $\leftrightarrow_{\mathcal{E}}^*$. Mark in each step which part of your term you are replacing and which equation you used for it.