Prof. Dr. Jürgen Giesl Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Friday, November 27th, 2015, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (4 + 2 = 6 points)

Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$.

$$\begin{aligned} \mathsf{b}(x) &\equiv \mathsf{e}(x) \\ \mathsf{a}(\mathsf{d}) &\equiv \mathsf{b}(x) \\ \mathsf{a}(x) &\equiv \mathsf{c}(\mathsf{a}(x)) \\ \mathsf{b}(\mathsf{d}) &\equiv \mathsf{c}(\mathsf{d}) \\ \mathsf{b}(\mathsf{b}(x)) &\equiv \mathsf{b}(\mathsf{c}(x)) \end{aligned}$$

a) Orient the equations from $\mathcal{E}$ such that the resulting TRS $\mathcal{R}$ is convergent (without proof). Briefly explain your choice for each equation.

Hint: There is one and only one solution resulting in a convergent TRS.

b) Decide the following equivalences using the algorithm WORD_PROBLEM.

$$\begin{aligned} \mathsf{c}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{e}(\mathsf{e}(\mathsf{b}(\mathsf{c}(\mathsf{d})))))))) &\equiv_{\mathcal{E}} \mathsf{c}(\mathsf{a}(\mathsf{b}(\mathsf{a}(\mathsf{e}(\mathsf{c}(\mathsf{a}(\mathsf{a}(\mathsf{d})))))))) \\ \mathsf{c}(\mathsf{a}(\mathsf{b}(\mathsf{a}(\mathsf{e}(\mathsf{e}(\mathsf{c}(\mathsf{c}(\mathsf{d})))))))) &\equiv_{\mathcal{E}} \mathsf{a}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{c}(\mathsf{d})))))))) \end{aligned}$$

Hint: Since all function symbols are at most unary, you can omit all parentheses in your solution without introducing ambiguities.

Exercise 2 (Noetherian Induction): (2 + 4 = 6 points)

Consider the following term rewrite system $\mathcal{R}$, which represents the well-known Ackermann function:

$$\mathsf{ack}(\mathcal{O}, m) \to \mathsf{s}(m) \qquad (1)$$
$$\mathsf{ack}(\mathsf{s}(n), \mathcal{O}) \to \mathsf{ack}(n, \mathsf{s}(\mathcal{O})) \qquad (2)$$
$$\mathsf{ack}(\mathsf{s}(n), \mathsf{s}(m)) \to \mathsf{ack}(n, \mathsf{ack}(\mathsf{s}(n), m)) \qquad (3)$$

The goal of this exercise is to prove by Noetherian induction that, given two natural numbers (encoded as terms), ack computes a natural number.

a) Choose a suitable induction relation $\succ \subseteq \{(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \mid n_1, k_1 \in \mathbb{N}\} \times \{(\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \mid n_2, k_2 \in \mathbb{N}\}$ and prove that it is well founded.

b) Prove that any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^m(\mathcal{O}))$ has the form $\mathsf{s}^\ell(\mathcal{O})$ by Noetherian induction using the relation $\succ$ from part a).

Exercise 3 (The Algorithm RIGHT_GROUND_TERMINATION): (3 + 2 = 5 points)

Prove or disprove termination of the following term rewrite systems over the signature $\Sigma = \{\mathsf{f}, \mathsf{g}, \mathsf{a}, \mathsf{b}\}$ using the algorithm RIGHT_GROUND_TERMINATION from the lecture:

a)

$$\begin{array}{rcl} \mathsf{f}(\mathsf{f}(x, y), z) & \to & \mathsf{f}(\mathsf{b}, \mathsf{f}(\mathsf{b}, \mathsf{a})) \\ \mathsf{f}(\mathsf{a}, \mathsf{f}(x, y)) & \to & \mathsf{f}(\mathsf{f}(\mathsf{b}, \mathsf{a}), \mathsf{a}) \\ \mathsf{f}(x, \mathsf{b}) & \to & \mathsf{f}(\mathsf{b}, \mathsf{a}) \\ \mathsf{f}(\mathsf{b}, x) & \to & \mathsf{b} \end{array}$$

b)

$$\begin{array}{rcl} \mathsf{f}(\mathsf{s}(x), \mathsf{g}(x)) & \to & \mathsf{g}(\mathsf{s}(\mathsf{a})) \\ \mathsf{f}(\mathsf{s}(x), \mathsf{s}(x)) & \to & \mathsf{g}(\mathsf{f}(\mathsf{s}(\mathsf{a}), \mathsf{g}(\mathsf{s}(\mathsf{a})))) \\ \mathsf{g}(\mathsf{s}(x)) & \to & \mathsf{s}(\mathsf{a}) \end{array}$$