Prof. Dr. Jürgen Giesl — Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on **Friday, December 4th, 2015**, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Reduction and Simplification Orders): (2 + 6 = 8 points)

a) The following TRS $\mathcal{R}$ is terminating. Please show that there is no simplification order by which termination of $\mathcal{R}$ can be proved, i.e., for every simplification order $\succ$ we have $\to_{\mathcal{R}} \not\subseteq \succ$.

$$\begin{array}{rcl} \mathsf{half}(\mathcal{O}) & \to & \mathcal{O} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) & \to & \mathsf{s}(\mathsf{half}(x)) \\ \mathsf{log}(\mathsf{s}(\mathcal{O})) & \to & \mathcal{O} \\ \mathsf{log}(\mathsf{s}(\mathsf{s}(x))) & \to & \mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x))))) \end{array}$$

b) Please prove or disprove the following propositions. Here, $\rhd$ denotes the subterm relation.

i For every well-founded relation $\succ$ we have that $\succ \cup \rhd$ is well-founded.

ii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is well-founded.

iii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is a reduction order.

iv For every reduction order $\succ$ we have that $\succ \cup \succ_{emb}$ is well-founded.

Hints:

- You may use the previous exercise part.

Exercise 2 (Kruskal's theorem): (3 points)

Prove or disprove: If a TRS $\mathcal{R}$ is non-terminating, then there are terms $s, t \in \mathcal{T}(\Sigma, \emptyset)$ such that $s \to_{\mathcal{R}}^{+} t$ and $s \not\succ t$ for all simplification orders $\succ$.

Exercise 3 (Termination Proofs with Simplification Orders): (1 + 3 + 2 = 6 points)

Please prove termination of the following TRSs using the embedding order. If this is not possible, use LPO or LPOS instead and explicitly state the precedence (and the status) you are using. In this exercise, x, y, xs, ys, z, and zs denote variables while all other identifiers denote function symbols.

To prove that for two terms t_1 and t_2 we have $t_1 \succ_{emb} t_2$, $t_1 \succ_{lpo} t_2$, or $t_1 \succ_{lpos} t_2$, use a proof tree notation to indicate which case of the definition of $\succ_{emb}$, $\succ_{lpo}$, or $\succ_{lpos}$ you are using. This is illustrated by the

following example where we have $t_1 = \mathsf{f}(\mathcal{O}, \mathsf{s}(x))$, $t_2 = \mathsf{f}(\mathsf{s}(\mathcal{O}), x)$, and $t_1 \succ_{lpos} t_2$:
Choose $\mathsf{f} \sqsupset \mathsf{s}$ and $\langle 2, 1 \rangle$ for f. Then we have

$$\dfrac{\dfrac{\overline{x \succeq_{lpos} x}\;=}{\mathsf{s}(x) \succ_{lpos} x}\;1 \qquad \dfrac{\dfrac{\overline{\mathcal{O} \succeq_{lpos} \mathcal{O}}\;=}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathcal{O}}\;1}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathsf{s}(\mathcal{O})}\;2}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathsf{f}(\mathsf{s}(\mathcal{O}), x)}\;3$$

Hint: You may abbreviate names of function symbols (e.g., "a" instead of "append").

a)

$$\begin{aligned} \mathsf{nth}(\mathcal{O}, \mathsf{Cons}(y, ys)) &\rightarrow y \\ \mathsf{nth}(\mathsf{s}(x), \mathsf{Cons}(y, ys)) &\rightarrow \mathsf{nth}(x, ys) \end{aligned}$$

b)

$$\begin{aligned} \mathsf{append}(\mathsf{Nil}, ys) &\rightarrow ys \\ \mathsf{append}(\mathsf{Cons}(x, xs), ys) &\rightarrow \mathsf{Cons}(x, \mathsf{append}(xs, ys)) \\ \mathsf{reverse}(\mathsf{Nil}) &\rightarrow \mathsf{Nil} \\ \mathsf{reverse}(\mathsf{Cons}(x, xs)) &\rightarrow \mathsf{append}(\mathsf{reverse}(xs), \mathsf{Cons}(x, \mathsf{Nil})) \end{aligned}$$

c)

$$\begin{aligned} \mathsf{sum}(x, \mathcal{O}, \mathsf{Nil}) &\rightarrow x \\ \mathsf{sum}(x, \mathsf{s}(y), zs) &\rightarrow \mathsf{sum}(\mathsf{s}(x), y, zs) \\ \mathsf{sum}(x, \mathcal{O}, \mathsf{Cons}(z, zs)) &\rightarrow \mathsf{sum}(x, z, zs) \end{aligned}$$