# Term Rewriting Systems

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# Exercise 1: Monotonicity

Prove or disprove monotonicity for each of the following relations:

- $s \sim_1 t$ iff $|s| = |t|$
- $s \sim_2 t$ iff $s$ matches $t$ and $t$ matches $s$

# Exercise 1: Monotonicity

Prove or disprove monotonicity of the relation $\sim_1$ defined by

$$s \sim_1 t \iff |s| = |t|$$

- To show: for all $s, t, q \in \mathcal{T}, \pi \in Occ(q)$ we have $|s| = |t| \implies |q[s]_\pi| = |q[t]_\pi|$
- Induction on $\pi$
- **IB** ($\pi = \varepsilon$): $q[s]_\pi = s$, $q[t]_\pi = t$, and hence $|q[s]_\pi| = |q[t]_\pi|$
- **IS** ($\pi = i\pi'$)
  - **IH**: for all $s', t', q' \in \mathcal{T}$ such that $|s'| = |t'|$ and $\pi' \in Occ(q')$ we have $|q'[s']_{\pi'}| = |q'[t']_{\pi'}|$
  - $q = f(q_1, \ldots, q_n)$, $n \geq i$
  - $|f(q_1, \ldots, q_n)[s]_{i\pi'}| = 1 + |q_1| + \cdots + |q_i[s]_{\pi'}| + \cdots + |q_n|$
  - $|f(q_1, \ldots, q_n)[t]_{i\pi'}| = 1 + |q_1| + \cdots + |q_i[t]_{\pi'}| + \cdots + |q_n|$
  - By **IH**, we have $|q_i[s]_{\pi'}| = |q_i[t]_{\pi'}|$

$\implies$ $|q[s]_{i\pi'}| = |q[t]_{i\pi'}|$

# Exercise 1: Monotonicity

Prove or disprove monotonicity of the relation $\sim_2$ defined by

$$s \sim_2 t \iff s \text{ matches } t \text{ and } t \text{ matches } s$$

- let $s = x$, $t = y$, $q = \mathsf{f}(x, y)$, and $\pi = 2$
- we have $s \sim_2 t$
- $q[s]_\pi = \mathsf{f}(x, x)$ does not match $q[t]_\pi = \mathsf{f}(x, y)$

# Exercise 2: Equivalence relations

Prove or disprove: $\sim_1$ with $s \sim_1 t$ iff $|s| = |t|$ is an equivalence relation.

- Let $t, s, q \in \mathcal{T}$
- **Reflexivity** we have $t \sim_1 t$, as $|t| = |t|$
- **Symmetry** if $t \sim_1 s$, then $|t| = |s|$ and consequently also $s \sim_1 t$
- **Transitivity** if $t \sim_1 s$ and $s \sim_1 q$, then $|t| = |s| = |q|$ and thus $t \sim_1 q$

# Exercise 2: Equivalence relations

Prove or disprove: $\sim_2$ with $s \sim_2 t$ iff $s$ matches $t$ and $t$ matches $s$ is an equivalence relation.

- Let $t, s, q \in \mathcal{T}$
- **Reflexivity** we have $t\sigma = t$ with $\sigma = \emptyset$
- **Symmetry**
  - if $s \sim_2 t$ then there are substitutions $\sigma, \theta$ such that $s\sigma = t$ and $t\theta = s$

  $\implies$ $t \sim_2 s$
- **Transitivity**
  - if $s \sim_2 t$, then there are substitutions $\sigma, \theta$ such that $s\sigma = t$ and $t\theta = s$
  - if $t \sim_2 q$, then there are substitutions $\sigma', \theta'$ such that $t\sigma' = q$ and $q\theta' = t$

  $\implies$ $s\sigma\sigma' = q$ and $q\theta'\theta = s$

# Exercise 2: Equivalence relations

Prove or disprove: For all $\mathcal{E}$, $\rightarrow^*_{\mathcal{E}}$ is an equivalence relation.

- let $\mathcal{E} = \{c \equiv d\}$
- then $c \rightarrow^*_{\mathcal{E}} d$, but $d \not\rightarrow^*_{\mathcal{E}} c$

$\implies$ $\rightarrow^*_{\mathcal{E}}$ is not symmetric

$\implies$ $\rightarrow^*_{\mathcal{E}}$ is not an equivalence relation

# Exercise 3: Equivalence classes

Let $|t|_\Sigma$ denote the number of function symbols in $t$ and let $s \sim t$ iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$. Show that $\sim$ is an equivalence relation and that all equivalence classes are finite.

- **Reflexivity** for each $t \in \mathcal{T}$, $\mathcal{V}(t) = \mathcal{V}(t)$ and $|t|_\Sigma = |t|_\Sigma$ holds
- **Symmetry**
  - $\mathcal{V}(t) = \mathcal{V}(s)$ implies $\mathcal{V}(s) = \mathcal{V}(t)$
  - $|t|_\Sigma = |s|_\Sigma$ implies $|s|_\Sigma = |t|_\Sigma$
- **Transitivity**
  - $\mathcal{V}(t) = \mathcal{V}(s) \wedge \mathcal{V}(s) = \mathcal{V}(q)$ implies $\mathcal{V}(t) = \mathcal{V}(q)$
  - $|t|_\Sigma = |s|_\Sigma \wedge |s|_\Sigma = |q|_\Sigma$ implies $|t|_\Sigma = |q|_\Sigma$

# Exercise 3: Equivalence classes

Let $s \sim t$ iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$. Show that all equivalence classes w.r.t. $\sim$ are finite.

**Short version:** There are only finitely many combinations of finitely many function symbols and variables.

# Exercise 3: Equivalence classes

Let $s \sim t$ iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$. Show that all equivalence classes w.r.t. $\sim$ are finite.

**Long version:**

- to show: $[s]_\sim$ is finite
- let $|s|_\Sigma = k$

$\Longrightarrow$ the longest position in $s$ has at most length $k+1$

- let $n$ be the maximal arity of function symbols in $\Sigma$

$\Longrightarrow$ each position in $s$ is from $\{1, \ldots, n\}^*$

$\Longrightarrow$ $s$ has at most $(n+1)^{k+1}$ positions

- at each position, there is either a variable or a function symbol

$\Longrightarrow$ there are at most $(n+1)^{k+1} \cdot |\Sigma| \cdot |\mathcal{V}(s)|$ terms in $[s]_\sim$

$\Longrightarrow$ $[s]_\sim$ is finite

# Exercise 3: Equivalence classes

Let $s \sim t$ iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$. Show that the word problem is decidable for the following set of equations $\mathcal{E}$.

$$\begin{aligned} \mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \\ \mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) \\ \mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) \end{aligned}$$

- $s \sim t$ holds for all equations $s \equiv t \in \mathcal{E}$

$\stackrel{?}{\Longrightarrow}$ for each term $q$, $[q]_{\equiv_\mathcal{E}} \subseteq [q]_\sim$

- No!
  - $\mathcal{E} = \{f(f(x, x), y) \equiv f(f(y, y), x)\}$
  - We have $f(f(x, x), y) \sim f(f(y, y), x)$...
  - and $f(f(a, a), g(a)) \equiv_\mathcal{E} f(f(g(a), g(a)), a)$
  - But: $f(f(a, a), g(a)) \not\sim f(f(g(a), g(a)), a)$
  - Problem: $\sim$ is not stable

# Exercise 3: Equivalence classes

Let $s \sim' t$ iff $|s|_x = |t|_x$ for all $x \in \mathcal{V}(s) \cup \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$.

- $\sim'$ is an equivalence relation. Proof is similar to $\sim$.
- $\sim'$ is stable
  - for all $x \in \mathcal{V}(s\sigma) \cup \mathcal{V}(t\sigma)$, we have
    $|s\sigma|_x = \sum_{y \in \mathcal{V}(s) \cup \mathcal{V}(t)} |s|_y \cdot |y\sigma|_x$ and
    $|t\sigma|_x = \sum_{y \in \mathcal{V}(s) \cup \mathcal{V}(t)} |t|_y \cdot |y\sigma|_x$
  - since we have $|s|_y = |t|_y$ for all $y \in \mathcal{V}(s) \cup \mathcal{V}(t)$, this implies
    $|s\sigma|_y = |t\sigma|_y$
  - $|s\sigma|_\Sigma = |t\sigma|_\Sigma$: similar
- $\sim'$ is monotonic
  - for all $x \in \mathcal{V}(s\sigma) \cup \mathcal{V}(t\sigma)$, we have
    $|q[s]_\pi|_x = |q|_x - |q|_\pi|_x + |s|_x$ and $|q[t]_\pi|_x = |q|_x - |q|_\pi|_x + |t|_x$
  - since we have $|s|_x = |t|_x$ for all $x \in \mathcal{V}(s) \cup \mathcal{V}(t)$, this implies
    $|q[s]_\pi|_x = |q[t]_\pi|_x$
  - $|q[s]_\pi|_\Sigma = |q[t]_\pi|_\Sigma$: similar

# Exercise 3: Equivalence classes

Let $s \sim' t$ iff $|s|_x = |t|_x$ for all $x \in \mathcal{V}(s) \cup \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$.

$$\begin{aligned} \mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \\ \mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) \\ \mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) \end{aligned}$$

- $s \sim' t$ holds for all equations $s \equiv t \in \mathcal{E}$

$\Longrightarrow$ for each term $q$, $[q]_{\equiv_\mathcal{E}} \subseteq [q]_{\sim'}$

$\Longrightarrow$ for each term $q$, $[q]_{\equiv_\mathcal{E}}$ is finite

$\Longrightarrow$ to decide $s \equiv_\mathcal{E} t$, compute $[s]_{\equiv_\mathcal{E}}$

# Exercise 4: Syntactic Proofs

Consider the following set of equations $\mathcal{E}$:

$$\begin{aligned}
\mathsf{union}(\mathsf{nil}, xs) &\equiv xs && (1)\\
\mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) && (2)\\
\mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) && (3)\\
\mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) && (4)\\
\mathsf{cons}(x, \mathsf{cons}(x, xs)) &\equiv \mathsf{cons}(x, xs) && (5)
\end{aligned}$$

Prove

$$\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil})) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))$$

using $\leftrightarrow^*_{\mathcal{E}}$.

# Exercise 4: Syntactic Proofs

$$\mathsf{union}(\mathsf{nil}, xs) \equiv xs \tag{1}$$
$$\mathsf{union}(\mathsf{cons}(x, xs), ys) \equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \tag{2}$$
$$\mathsf{union}(xs, ys) \equiv \mathsf{union}(ys, xs) \tag{3}$$
$$\mathsf{cons}(x, \mathsf{cons}(y, ys)) \equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) \tag{4}$$
$$\mathsf{cons}(x, \mathsf{cons}(x, xs)) \equiv \mathsf{cons}(x, xs) \tag{5}$$

To prove: $\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil})) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))$

$$\begin{aligned}
& \underline{\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil}))} \\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}} \; & \mathsf{cons}(x, \underline{\mathsf{union}(\mathsf{cons}(y, \mathsf{nil}), \mathsf{cons}(x, \mathsf{nil}))}) \\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}} \; & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{union}(\mathsf{nil}, \mathsf{cons}(x, \mathsf{nil}))})) \\
\overset{(1)}{\leftrightarrow}_{\mathcal{E}} \; & \mathsf{cons}(x, \underline{\mathsf{cons}(y, \mathsf{cons}(x, \mathsf{nil}))}) \\
\overset{(4)}{\leftrightarrow}_{\mathcal{E}} \; & \underline{\mathsf{cons}(x, \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})))} \\
\overset{(5)}{\leftrightarrow}_{\mathcal{E}} \; & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))
\end{aligned}$$

# Exercise 4: Syntactic Proofs

$$\begin{aligned}
\mathrm{union}(\mathrm{nil}, xs) &\equiv xs && (1)\\
\mathrm{union}(\mathrm{cons}(x, xs), ys) &\equiv \mathrm{cons}(x, \mathrm{union}(xs, ys)) && (2)\\
\mathrm{union}(xs, ys) &\equiv \mathrm{union}(ys, xs) && (3)\\
\mathrm{cons}(x, \mathrm{cons}(y, ys)) &\equiv \mathrm{cons}(y, \mathrm{cons}(x, ys)) && (4)\\
\mathrm{cons}(x, \mathrm{cons}(x, xs)) &\equiv \mathrm{cons}(x, xs) && (5)
\end{aligned}$$

$$\mathrm{union}(\mathrm{cons}(x, \mathrm{cons}(y, xs)), \mathrm{cons}(z, \mathrm{cons}(y, ys))) \equiv_{\mathcal{E}} \mathrm{cons}(x, \mathrm{cons}(y, \mathrm{cons}(z, \mathrm{union}(xs, ys))))$$

$$\begin{aligned}
&\underline{\mathrm{union}(\mathrm{cons}(x, \mathrm{cons}(y, xs)), \mathrm{cons}(z, \mathrm{cons}(y, ys)))}\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ &\mathrm{cons}(x, \underline{\mathrm{union}(\mathrm{cons}(y, xs), \mathrm{cons}(z, \mathrm{cons}(y, ys)))})\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ &\mathrm{cons}(x, \mathrm{cons}(y, \underline{\mathrm{union}(xs, \mathrm{cons}(z, \mathrm{cons}(y, ys)))}))\\
\overset{(3)}{\leftrightarrow}_{\mathcal{E}}\ &\mathrm{cons}(x, \mathrm{cons}(y, \underline{\mathrm{union}(\mathrm{cons}(z, \mathrm{cons}(y, ys)), xs)}))\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ &\mathrm{cons}(x, \mathrm{cons}(y, \mathrm{cons}(z, \mathrm{union}(\mathrm{cons}(y, ys), xs))))
\end{aligned}$$

# Exercise 4: Syntactic Proofs

$$\mathsf{union}(\mathsf{nil}, xs) \equiv xs \quad (1)$$
$$\mathsf{union}(\mathsf{cons}(x, xs), ys) \equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \quad (2)$$
$$\mathsf{union}(xs, ys) \equiv \mathsf{union}(ys, xs) \quad (3)$$
$$\mathsf{cons}(x, \mathsf{cons}(y, ys)) \equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) \quad (4)$$
$$\mathsf{cons}(x, \mathsf{cons}(x, xs)) \equiv \mathsf{cons}(x, xs) \quad (5)$$

$$\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, xs)), \mathsf{cons}(z, \mathsf{cons}(y, ys))) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(xs, ys))))$$

$$\begin{aligned}
& \mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, xs)), \mathsf{cons}(z, \mathsf{cons}(y, ys))) \\
\leftrightarrow^*_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \underline{\mathsf{union}(\mathsf{cons}(y, ys), xs)}))) \\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{cons}(z, \mathsf{cons}(y, \mathsf{union}(ys, xs))))})) \\
\overset{(4)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \underline{\mathsf{cons}(y, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(ys, xs)))))}) \\
\overset{(5)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \underline{\mathsf{union}(ys, xs)}))) \\
\overset{(3)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(xs, ys))))
\end{aligned}$$