

Term Rewriting Systems

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Exercise 1

- ▶ $s \rightarrow^{\leq k} t, k \in \mathbb{N}$, iff there is a $c \leq k$ such that $s \rightarrow^c t$
- ▶ \rightarrow is *k-confluent* iff $\exists k \in \mathbb{N} \setminus \{0\}$ such that $p \rightarrow^{\leq k} s$ and $p \rightarrow^{\leq k} t$ implies $\exists q$ with $s \rightarrow^{\leq k} q$ and $t \rightarrow^{\leq k} q$.

Prove or disprove: *k*-confluence implies confluence.

Exercise 2

\rightarrow is *semi-confluent* iff $p \rightarrow s$ and $p \rightarrow^* t$ implies $\exists q$ such that $s \rightarrow^* q$ and $t \rightarrow^* q$.

Prove or disprove: semi-confluence implies confluence.

Exercise 3

$qs(\text{Nil})$	\rightarrow	Nil
$qs(\text{Cons}(x, xs))$	\rightarrow	$\text{app}(qs(l(x, xs)), \text{Cons}(x, qs(h(x, xs))))$
$l(x, \text{Nil})$	\rightarrow	Nil
$l(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifl}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifl}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$l(x, ys)$
$\text{ifl}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, l(x, ys))$
$h(x, \text{Nil})$	\rightarrow	Nil
$h(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifh}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifh}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, h(x, ys))$
$\text{ifh}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$h(x, ys)$
$\text{leq}(\mathcal{O}, x)$	\rightarrow	\top
$\text{leq}(s(x), \mathcal{O})$	\rightarrow	\perp
$\text{leq}(s(x), s(y))$	\rightarrow	$\text{leq}(x, y)$
$\text{app}(\text{Nil}, ys)$	\rightarrow	ys
$\text{app}(\text{Cons}(x, xs), ys)$	\rightarrow	$\text{Cons}(x, \text{app}(xs, ys))$

Prove or disprove: \mathcal{R}_{qs} is confluent.

Solution: \mathcal{R}_{qs} is orthogonal and hence confluent.

Exercise 3

$qs(\text{Nil})$	\rightarrow	Nil
$qs(\text{Cons}(x, xs))$	\rightarrow	$\text{app}(qs(l(x, xs)), \text{Cons}(x, qs(h(x, xs))))$
$l(x, \text{Nil})$	\rightarrow	Nil
$l(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifl}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifl}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$l(x, ys)$
$\text{ifl}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, l(x, ys))$
$h(x, \text{Nil})$	\rightarrow	Nil
$h(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifh}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifh}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, h(x, ys))$
$\text{ifh}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$h(x, ys)$
$\text{leq}(\mathcal{O}, x)$	\rightarrow	\top
$\text{leq}(s(x), \mathcal{O})$	\rightarrow	\perp
$\text{leq}(s(x), s(y))$	\rightarrow	$\text{leq}(x, y)$
$\text{app}(\text{Nil}, ys)$	\rightarrow	ys
$\text{app}(\text{Cons}(x, xs), ys)$	\rightarrow	$\text{Cons}(x, \text{app}(xs, ys))$

Prove or disprove: \mathcal{R}_{qs} is confluent.

Solution: \mathcal{R}_{qs} is orthogonal and hence confluent.

Exercise 3

$qs(\text{Nil})$	\rightarrow	Nil
$qs(\text{Cons}(x, xs))$	\rightarrow	$\text{app}(qs(l(x, xs)), \text{Cons}(x, qs(h(x, xs))))$
$l(x, \text{Nil})$	\rightarrow	Nil
$l(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifl}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifl}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$l(x, ys)$
$\text{ifl}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, l(x, ys))$
$h(x, \text{Nil})$	\rightarrow	Nil
$h(x, \text{Cons}(y, ys))$	\rightarrow	$\text{ifh}(\text{leq}(x, y), x, \text{Cons}(y, ys))$
$\text{ifh}(\top, x, \text{Cons}(y, ys))$	\rightarrow	$\text{Cons}(y, h(x, ys))$
$\text{ifh}(\perp, x, \text{Cons}(y, ys))$	\rightarrow	$h(x, ys)$
$\text{leq}(\mathcal{O}, x)$	\rightarrow	\top
$\text{leq}(s(x), \mathcal{O})$	\rightarrow	\perp
$\text{leq}(s(x), s(y))$	\rightarrow	$\text{leq}(x, y)$
$\text{app}(\text{Nil}, ys)$	\rightarrow	ys
$\text{app}(\text{Cons}(x, xs), ys)$	\rightarrow	$\text{Cons}(x, \text{app}(xs, ys))$

Normalize $qs(\text{Cons}(s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil})))$ w.r.t. $\Rightarrow_{\mathcal{R}_{qs}}$.

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\text{qs}(\underline{\text{l}(s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))}), \text{Cons}(s(\mathcal{O}), \text{qs}(\underline{\text{h}(s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))})))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\text{qs}(\underline{\text{ifl}(\underline{\text{leq}(s(\mathcal{O}), \mathcal{O})}, s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))}), \text{Cons}(s(\mathcal{O}), \text{qs}(\underline{\text{ifh}(\underline{\text{leq}(s(\mathcal{O}), \mathcal{O})}, s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))})))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\text{qs}(\underline{\text{ifl}(\overline{\mathbb{1}}, s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))}), \text{Cons}(s(\mathcal{O}), \text{qs}(\underline{\text{ifh}(\overline{\mathbb{1}}, s(\mathcal{O}), \text{Cons}(\mathcal{O}, \text{Nil}))})))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\text{qs}(\underline{\text{Cons}(\mathcal{O}, \underline{\text{l}(s(\mathcal{O}), \text{Nil}))}), \text{Cons}(s(\mathcal{O}), \text{qs}(\underline{\text{h}(s(\mathcal{O}), \text{Nil}))})))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\text{qs}(\underline{\text{Cons}(\mathcal{O}, \overline{\text{Nil}})}), \text{Cons}(s(\mathcal{O}), \text{qs}(\overline{\text{Nil}}))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\overline{\text{app}(\text{qs}(\underline{\text{l}(\mathcal{O}, \text{Nil}))}), \text{Cons}(\mathcal{O}, \text{qs}(\underline{\text{h}(\mathcal{O}, \text{Nil}))}))), \text{Cons}(s(\mathcal{O}), \overline{\text{Nil}})}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\overline{\text{app}(\text{qs}(\overline{\text{Nil}}), \text{Cons}(\mathcal{O}, \text{qs}(\overline{\text{Nil}}))}), \text{Cons}(s(\mathcal{O}), \text{Nil}))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\overline{\text{app}(\overline{\text{Nil}}, \text{Cons}(\mathcal{O}, \overline{\text{Nil}})}), \text{Cons}(s(\mathcal{O}), \text{Nil}))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{app}(\overline{\text{Cons}(\mathcal{O}, \text{Nil})}, \text{Cons}(s(\mathcal{O}), \text{Nil}))}$

$\Rightarrow_{\mathcal{R}_{qs}} \overline{\text{Cons}(\mathcal{O}, \overline{\text{app}(\text{Nil}, \text{Cons}(s(\mathcal{O}), \text{Nil}))})} \Rightarrow_{\mathcal{R}_{qs}} \overline{\text{Cons}(\mathcal{O}, \overline{\text{Cons}(s(\mathcal{O}), \text{Nil}))}}$

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(\perp, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(\perp, s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil)))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(Nil), Cons(O, qs(Nil))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(Nil, Cons(O, Nil)), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(Cons(O, Nil), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl($\bar{1}$, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh($\bar{1}$, s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil)))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(Nil), Cons(O, qs(Nil))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(Nil, Cons(O, Nil)), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(Cons(O, Nil), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O)), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O)), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥), s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O)), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifeh(leq(s(O), O)), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥), s(O), Cons(O, Nil))), Cons(s(O), qs(ifeh(⊥), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

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Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil))))))

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Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil)))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil)))))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil))))))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil)))))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil)))), Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil)))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥, s(O), Cons(O, Nil)))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil)))), Cons(s(O), Nil)

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(Nil), Cons(O, qs(Nil))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(Nil, Cons(O, Nil)), Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O)), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O)), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥), s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil))))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(Nil), Cons(O, qs(Nil))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(Nil, Cons(O, Nil)), Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(\perp, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(\perp, s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil)))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(Nil), Cons(O, qs(Nil))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(Nil, Cons(O, Nil)), Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O)), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O)), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(⊥), s(O), Cons(O, Nil))), Cons(s(O), qs(ifh(⊥), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, Nil)), Cons(s(O), qs(Nil)))

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$\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, app(Nil, Cons(s(O), Nil))) $\Rightarrow_{\mathcal{R}_{qs}}$ Cons(O, Cons(s(O), Nil))

qs(Cons(s(O), Cons(O, Nil)))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(l(s(O), Cons(O, Nil))), Cons(s(O), qs(h(s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl(leq(s(O), O)), s(O), Cons(O, Nil))),
Cons(s(O), qs(ifh(leq(s(O), O)), s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(ifl($\bar{\perp}$, s(O), Cons(O, Nil))), Cons(s(O), qs(ifh($\bar{\perp}$, s(O), Cons(O, Nil))))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, l(s(O), Nil))), Cons(s(O), qs(h(s(O), Nil))))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(qs(Cons(O, $\bar{\text{Nil}}$)), Cons(s(O), qs($\bar{\text{Nil}}$))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil))))), Cons(s(O), $\bar{\text{Nil}}$)

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs($\bar{\text{Nil}}$), Cons(O, qs($\bar{\text{Nil}}$))), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(app($\bar{\text{Nil}}$, Cons(O, $\bar{\text{Nil}}$)), Cons(s(O), Nil))

$\Rightarrow_{\mathcal{R}_{qs}}$ app(Cons(O, Nil), Cons(s(O), Nil))

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$\Rightarrow_{\mathcal{R}_{qs}}$ app(app(qs(l(O, Nil)), Cons(O, qs(h(O, Nil))))), Cons(s(O), Nil)

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Exercise 4

Try to use BASIC_COMPLETION to complete the following TRSs:

$$\text{element}(\text{Cons}(x, xs)) \rightarrow x \quad (1)$$

$$\text{element}(\text{Cons}(x, xs)) \rightarrow \text{element}(xs) \quad (2)$$

reduction order: LPO with precedence $\text{element} \sqsupset \text{Cons}$

$$f(x) \rightarrow s(\text{pred}(x)) \quad (1)$$

$$f(x) \rightarrow \text{pred}(s(x)) \quad (2)$$

$$\text{pred}(s(x)) \rightarrow x \quad (3)$$

reduction order: LPO with precedence $f \sqsupset s \sqsupset \text{pred}$

$$f(f(x)) \rightarrow h(x) \quad (1)$$

$$f(g(x)) \rightarrow f(x) \quad (2)$$

$$f(x) \rightarrow g(x) \quad (3)$$

reduction order: LPO with precedence $f \sqsupset h \sqsupset g$.

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Precedence: $f \sqsupset h \sqsupset g$

Rules	Critical pair	normal forms	new rule
1,1	$\langle f(h(x)), h(f(x)) \rangle$	$h(g(x)), g(h(x))$	$h(g(x)) \rightarrow g(h(x)) \quad (4)$
1,2	$\langle h(g(x)), f(f(x)) \rangle$	$h(g(x)), h(x)$	$h(g(x)) \rightarrow h(x) \quad (5)$
1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
1,3	$\langle h(x), g(f(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
2,3	$\langle f(x), g(g(x)) \rangle$	$g(x), g(g(x))$	$g(g(x)) \rightarrow g(x) \quad (7)$
2,7	$\langle f(g(x)), f(g(x)) \rangle$	$g(x), g(x)$	
4,5	$\langle g(h(x)), h(x) \rangle$	$g(x), g(x)$	
4,6	$\langle g(h(x)), g(g(g(x))) \rangle$	$g(x), g(x)$	
4,7	$\langle g(h(g(x))), h(g(x)) \rangle$	$g(x), g(x)$	
5,6	$\langle h(x), g(g(g(x))) \rangle$	$g(x), g(x)$	
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1,1	$\langle f(h(x)), h(f(x)) \rangle$	$h(g(x)), g(h(x))$	$h(g(x)) \rightarrow g(h(x)) \quad (4)$
1,2	$\langle h(g(x)), f(f(x)) \rangle$	$h(g(x)), h(x)$	$h(g(x)) \rightarrow h(x) \quad (5)$
1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
1,3	$\langle h(x), g(f(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
2,3	$\langle f(x), g(g(x)) \rangle$	$g(x), g(g(x))$	$g(g(x)) \rightarrow g(x) \quad (7)$
2,7	$\langle f(g(x)), f(g(x)) \rangle$	$g(x), g(x)$	
4,5	$\langle g(h(x)), h(x) \rangle$	$g(x), g(x)$	
4,6	$\langle g(h(x)), g(g(g(x))) \rangle$	$g(x), g(x)$	
4,7	$\langle g(h(g(x))), h(g(x)) \rangle$	$g(x), g(x)$	
5,6	$\langle h(x), g(g(g(x))) \rangle$	$g(x), g(x)$	
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1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
1,3	$\langle h(x), g(f(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
2,3	$\langle f(x), g(g(x)) \rangle$	$g(x), g(g(x))$	$g(g(x)) \rightarrow g(x) \quad (7)$
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1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
1,3	$\langle h(x), g(f(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
2,3	$\langle f(x), g(g(x)) \rangle$	$g(x), g(g(x))$	$g(g(x)) \rightarrow g(x) \quad (7)$
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1,2	$\langle h(g(x)), f(f(x)) \rangle$	$h(g(x)), h(x)$	$h(g(x)) \rightarrow h(x) \quad (5)$
1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
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Precedence: $f \sqsupset h \sqsupset g$

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1,1	$\langle f(h(x)), h(f(x)) \rangle$	$h(g(x)), g(h(x))$	$h(g(x)) \rightarrow g(h(x)) \quad (4)$
1,2	$\langle h(g(x)), f(f(x)) \rangle$	$h(g(x)), h(x)$	$h(g(x)) \rightarrow h(x) \quad (5)$
1,3	$\langle h(x), f(g(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
1,3	$\langle h(x), g(f(x)) \rangle$	$h(x), g(g(x))$	$h(x) \rightarrow g(g(x)) \quad (6)$
2,3	$\langle f(x), g(g(x)) \rangle$	$g(x), g(g(x))$	$g(g(x)) \rightarrow g(x) \quad (7)$
2,7	$\langle f(g(x)), f(g(x)) \rangle$	$g(x), g(x)$	
4,5	$\langle g(h(x)), h(x) \rangle$	$g(x), g(x)$	
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