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Exercise 1 (Syntax and Semantics): (2 + 4 = 6 points)

a) Give a set of equalities that describes the functions $\mathsf{ge} : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$ and $\mathsf{odd} : \mathbb{N} \to \mathbb{B}$ with $\mathbb{B} = \{\top, \bot\}$ whose exact semantics should be as follows:

$$\mathsf{ge}(x, y) = \begin{cases} \top & \text{if } x \geq y \\ \bot & \text{otherwise} \end{cases}$$

$$\mathsf{odd}(x) = \begin{cases} \top & \text{if } x \text{ is odd} \\ \bot & \text{otherwise} \end{cases}$$

Thus, $\mathsf{odd}(15) = \top$, $\mathsf{odd}(0) = \bot$, $\mathsf{ge}(5, 10) = \bot$, and $\mathsf{ge}(5, 5) = \top$.

Use the representation of natural numbers presented in the lecture, where 0 is represented by $\mathcal{O} \in \Sigma_0$ and n is represented by applying a successor symbol $\mathsf{s} \in \Sigma_1$ n times (i.e., by $\mathsf{s}^n(\mathcal{O})$). Furthermore, use symbols $\mathsf{true}, \mathsf{false} \in \Sigma_0$ to represent $\top$ resp. $\bot$.

b) Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$, and $\Sigma_2 = \{\mathsf{plus}\}$. Consider $\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$, the set of equations describing addition on our representation of natural numbers.

Prove that $\mathcal{E} \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$.

Hints:

- You can use a model $A = (\mathcal{A}, \alpha)$ where $\mathcal{A}$ does not only consist of $\mathbb{N}$, but also contains additional elements (e.g., all words over some alphabet Π). Then define $\alpha_{\mathsf{plus}}(n, m)$ such that it models addition for $n, m \in \mathbb{N}$, but behaves differently if n or m are not from $\mathbb{N}$ (e.g., such that $\alpha_{\mathsf{plus}}(n, m)$ models concatenation if $n, m \in \Pi^*$).

Solution:

a) $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}, \mathsf{true}, \mathsf{false}\}$, $\Sigma_1 = \{\mathsf{s}, \mathsf{odd}\}$ and $\Sigma_2 = \{\mathsf{ge}\}$.

$$\begin{aligned} \mathsf{ge}(x, \mathcal{O}) &\equiv \mathsf{true} \\ \mathsf{ge}(\mathcal{O}, \mathsf{s}(y)) &\equiv \mathsf{false} \\ \mathsf{ge}(\mathsf{s}(x), \mathsf{s}(y)) &\equiv \mathsf{ge}(x, y) \end{aligned}$$

$$\begin{aligned} \mathsf{odd}(\mathsf{s}(\mathsf{s}(x))) &\equiv \mathsf{odd}(x) \\ \mathsf{odd}(\mathsf{s}(\mathcal{O})) &\equiv \mathsf{true} \\ \mathsf{odd}(\mathcal{O}) &\equiv \mathsf{false} \end{aligned}$$

b) As counterexample, consider $A = (\mathbb{N} \cup \{a, b\}^* \cup \{\bot\}, \alpha)$ where $\{a, b\}^*$ is the set of all finite words over the alphabet $\{a, b\}$ and

$$\alpha_{\mathcal{O}} := 0$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{plus}}(n, m) := \begin{cases} n+m & \text{if } n, m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ \bot & \text{otherwise} \end{cases}$$

Here, $\circ$ denotes concatenation.

To prove $A \models \mathcal{E}$, we consider each of the equations from $\mathcal{E}$. For the first one, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathcal{O}}, y) = \alpha_{\mathsf{plus}}(0, y) = y$. For the second equation, we have to consider five cases.

Case 1: $x, y \in \mathbb{N}$. Then for the left-hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}(x+1, y) = x + 1 + y$. For the right-hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(x + y) = 1 + x + y$.

Case 2: $x, y \in \{a, b\}^*$. Then for the left-hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}(\bot, y) = \bot$. For the right-hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(x \circ y) = \bot$.

Case 3: $x = 0$, $y \notin \mathbb{N}$. Then for the left-hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(0), y) = \alpha_{\mathsf{plus}}(1, y) = \bot$. For the right-hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(0, y)) = \alpha_{\mathsf{s}}(y) = \bot$.

Case 4: $x \in \mathbb{N} \setminus \{0\}, y \notin \mathbb{N}$. Then for the left-hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}(x+1, y) = \bot$. For the right-hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(\bot) = \bot$.

Case 5: $x \notin \mathbb{N}, \{x, y\} \not\subseteq \{a, b\}^*$. Then for the left-hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}(\bot, y) = \bot$. For the right-hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(\bot) = \bot$.

Thus, $A \models \{\mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$. We now show $A \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$. For the latter, consider the interpretation $I := (\mathbb{N} \cup \{\pi | \pi \in \{a, b\}^*\} \cup \{\bot\}, \alpha, \beta)$ with $\beta(x) = ab$ and $\beta(y) = ba$. Then $I(\mathsf{plus}(x, y)) = \alpha_{\mathsf{plus}}(ab, ba) = abba$ and $I(\mathsf{plus}(y, x)) = \alpha_{\mathsf{plus}}(ba, ab) = baab$.

Exercise 2 (Matching): (2 + 3 = 5 points)

a) Consider the following pairs of terms s and t over the signature $\Sigma = \Sigma_0 \cup \Sigma_2$ with $\Sigma_0 = \{\mathsf{a}\}$ and $\Sigma_2 = \{\mathsf{f}\}$. Moreover, we have $\{x, y, z\} \subseteq \mathcal{V}$ for the set of variables $\mathcal{V}$. If s matches t, then give a suitable matcher σ. Otherwise give a brief (at most two sentences) explanation why there is no matcher.

1. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, \mathsf{a})$
2. $s = \mathsf{f}(y, \mathsf{a})$, $t = \mathsf{f}(\mathsf{a}, x)$
3. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, x)$
4. $s = \mathsf{f}(x, y)$, $t = \mathsf{f}(\mathsf{f}(x, z), \mathsf{f}(x, z))$

b) Let $\sim$ be the matching relation, i.e., for two terms s and t we have $s \sim t$ iff s matches t.

Prove or disprove the following propositions:

1. For all terms s, t, and q we have $s \sim t \wedge t \sim q \implies s \sim q$.
2. For all terms s and t we have $s \neq t \wedge \mathcal{V}(s) = \mathcal{V}(t) \implies s \not\sim t$.

Solution:

a) 1. $\sigma = \{y/\mathsf{a}\}$

2. The term $\mathsf{f}(y, \mathsf{a})$ does not match the term $\mathsf{f}(\mathsf{a}, x)$, because a cannot be replaced by x using a substitution.
3. The term $\mathsf{f}(y, y)$ does not match the term $\mathsf{f}(\mathsf{a}, x)$, because y cannot be replaced by a and x at the same time using a substitution.
4. $\sigma = \{x/\mathsf{f}(x, z), y/\mathsf{f}(x, z)\}$

b) 1. The proposition is true: $s \sim t$ implies that there is a substitution σ such that $s\sigma = t$. Analogously, $t \sim q$ implies that there is a substitution θ such that $t\theta = q$. Hence we get $s\sigma\theta = t\theta = q$, i.e., $\sigma\theta$ is a matcher for s and q. Hence, we have $s \sim q$.

2. The proposition is wrong. Consider the terms $s = x$ and $t = \mathsf{f}(x)$. Then for $\sigma = \{x/\mathsf{f}(x)\}$ we have $s\sigma = t$.

Exercise 3 (Induction): (3 points)

Let $t \in \mathcal{T}(\Sigma, \mathcal{V})$, $\pi \in Occ(t)$, and $\sigma \in SUB(\Sigma, \mathcal{V})$. Show by induction over π that $(t|_\pi)\sigma = (t\sigma)|_\pi$ holds.

Hints:

- In the induction base, prove the proposition for $\pi = \epsilon$.
- In the induction step, consider the case $\pi = i\,\pi'$, where as induction hypothesis, you can assume that $(q|_{\pi'})\mu = (q\mu)|_{\pi'}$ for all $q \in \mathcal{T}(\Sigma, \mathcal{V})$ and all $\mu \in SUB(\Sigma, \mathcal{V})$.

Solution:

First, we consider the case $\pi = \epsilon$. Then, $t\sigma|_\pi = t\sigma = (t|_\pi)\sigma$.
Now let $\pi = i\,\pi'$ and assume that the proposition holds for π'. As $\pi \in Occ(t)$, we have $t = \mathsf{f}(q_1, \dots, q_i, \dots, q_n)$. By definition, $(t|_{i\,\pi'})\sigma = (q_i|_{\pi'})\sigma$ and $(q_i\sigma)|_{\pi'} = (t\sigma)|_{i\,\pi'}$ hold. By our induction hypothesis, we have $(q_i|_{\pi'})\sigma = (q_i\sigma)|_{\pi'}$ and thus, $(t|_{i\,\pi'})\sigma = (t\sigma)|_{i\,\pi'}$. □

Exercise 4 (Stability): (1 + 1 + 2 = 4 points)

Consider the following relations $\sim_1, \dots, \sim_3 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are stable.

a) $s \sim_1 t$ iff $t \trianglerighteq s$ (i.e., iff s is a subterm of t)

b) $s \sim_2 t$ iff s matches t

c) $s \sim_3 t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$

Hints:

- You can use the lemma proven in Exercise 3.

Solution:

a) $\sim_1$ is stable. Let s and t be terms with $t \trianglerighteq s$, i.e., there is a position π such that $t|_\pi = s$. Moreover, let σ be a substitution. Then we obtain $t\sigma|_\pi \overset{Hint}{=} t|_\pi\sigma \overset{t|_\pi = s}{=} s\sigma$ and, thus, $s\sigma$ is a subterm of $t\sigma$, i.e., $s\sigma \sim_1 t\sigma$.

b) $\sim_2$ is not stable. Consider the terms $s = \mathsf{f}(x)$ and $t = \mathsf{f}(y)$. We have $s\mu = t$ for the matcher $\mu = \{x/y\}$. Applying the substitution $\sigma = \{x/\mathsf{a}\}$ to both terms yields the terms $s\sigma = \mathsf{f}(\mathsf{a})$ and $t\sigma = \mathsf{f}(y)$ where $s\sigma$ does not match $t\sigma$ anymore.

c) $\sim_3$ is stable. Let s and t be terms with $\mathcal{V}(s) \subseteq \mathcal{V}(t)$ and σ be a substitution. Furthermore, let $x \in \mathcal{V}(s\sigma)$. If $x \notin \mathcal{V}(s)$ or $x \in DOM(\sigma)$, then there must be some $y \in \mathcal{V}(s)$ with $x \in \mathcal{V}(y\sigma)$. As we have $y \in \mathcal{V}(t)$, we also obtain $x \in \mathcal{V}(t\sigma)$. If $x \in \mathcal{V}(s) \setminus DOM(\sigma)$, then we obtain $x \in \mathcal{V}(t\sigma)$ again, because $x = x\sigma$ and $x \in \mathcal{V}(t)$.