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Exercise 1 (Reduction and Simplification Orders): (2 + 6 = 8 points)

a) The following TRS $\mathcal{R}$ is terminating. Please show that there is no simplification order by which termination of $\mathcal{R}$ can be proved, i.e., for every simplification order $\succ$ we have $\to_{\mathcal{R}} \not\subseteq \succ$.

$$\begin{array}{rcl} \mathsf{half}(\mathcal{O}) & \to & \mathcal{O} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) & \to & \mathsf{s}(\mathsf{half}(x)) \\ \mathsf{log}(\mathsf{s}(\mathcal{O})) & \to & \mathcal{O} \\ \mathsf{log}(\mathsf{s}(\mathsf{s}(x))) & \to & \mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x))))) \end{array}$$

b) Please prove or disprove the following propositions. Here, $\rhd$ denotes the subterm relation.

i For every well-founded relation $\succ$ we have that $\succ \cup \rhd$ is well-founded.

ii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is well-founded.

iii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is a reduction order.

iv For every reduction order $\succ$ we have that $\succ \cup \succ_{emb}$ is well-founded.

Hints:

- You may use the previous exercise part.

Solution:

a) Consider the rule $\mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x)))))$. We have $\mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x))))) \succ_{emb} \mathsf{log}(\mathsf{s}(\mathsf{s}(x)))$. Thus, every simplification order $\succ$ must also satisfy $\mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x))))) \succ \mathsf{log}(\mathsf{s}(\mathsf{s}(x)))$. Since every simplification order is well-founded, we obtain $\mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \not\succ \mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x)))))$.

b) i Wrong. Consider the relation $\succ = \{(\mathsf{a}, \mathsf{f}(\mathsf{a}))\}$. The relation $\succ$ is obviously well-founded, but we have $\mathsf{a} \succ \mathsf{f}(\mathsf{a}) \rhd \mathsf{a} \dots$ in contradiction to the well-foundedness of $\succ \cup \rhd$.

ii Right. Let $\succ$ be a reduction order. Assume $\succ \cup \rhd$ is not well-founded. Then there is an infinite sequence

$$t_{1,1} \succ t_{1,2} \succ \dots \succ t_{1,n_1} \rhd t_{2,1} \succ t_{2,2} \succ \dots \succ t_{2,n_2} \rhd t_{3,1} \dots$$

(since both $\succ$ and $\rhd$ are well-founded, they must both occur infinitely often in this sequence). Because of monotonicity of $\succ$ we obtain

$$t_{1,1} \succ t_{1,2} \succ \dots \succ t_{1,n_1} = q_1[t_{2,1}]_{\pi_1} \succ q_1[t_{2,2}]_{\pi_1} \succ \dots \succ q_1[t_{2,n_2}]_{\pi_1} = q_2[q_1[t_{3,1}]_{\pi_1}]_{\pi_2} \dots$$

and have, thus, a contradiction to the well-foundedness of $\succ$.

iii Wrong. The relation $\succ = \emptyset$ is a reduction order, but $\succ \cup \rhd = \rhd$ is no reduction order, since $\rhd$ is not monotonic.

iv Wrong. The TRS $\mathcal{R}$ from exercise part a) is terminating and, hence, $\to_{\mathcal{R}}$ is a reduction relation. Thus, $\succ = \to_{\mathcal{R}}^+$ is a reduction order. However, $\succ \cup \succ_{emb}$ is not well-founded, because we have

$$\mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{log}(\mathsf{half}(\mathsf{s}(\mathsf{s}(x))))) \succ_{emb} \mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \dots$$

Exercise 2 (Kruskal's theorem): (3 points)

Prove or disprove: If a TRS $\mathcal{R}$ is non-terminating, then there are terms $s, t \in \mathcal{T}(\Sigma, \emptyset)$ such that $s \to_{\mathcal{R}}^{+} t$ and $s \not\succ t$ for all simplification orders $\succ$.

Solution:

Since $\mathcal{R}$ is non-terminating, there is a term s_1 such that there is an infinite sequence $s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \ldots$.
Since $\to_{\mathcal{R}}$ is stable, each substitution $\sigma : \mathcal{V}(s_1) \to \mathcal{T}(\Sigma, \emptyset)$ gives rise to an infinite sequence $s_1\sigma = t_1 \to_{\mathcal{R}} s_2\sigma = t_2 \to_{\mathcal{R}} \ldots$ where t_1 is ground.
By definition of TRSs, we have $\mathcal{V}(r) \subseteq \mathcal{V}(\ell)$ for each $\ell \to r \in \mathcal{R}$. Hence, $t_2, t_3, \ldots$ are ground, too.
Since $t_1, t_2, \ldots$ is an infinite sequence of ground terms, there are $i, j \in \mathbb{N}$ such that $t_i \preceq_{emb} t_j$ by Kruskal's Theorem.
If $t_i = t_j$, then $t_i \not\succ t_j$ for each simplification order $\succ$, since each simplification order is well founded.
If $t_i \prec_{emb} t_j$, then we get $t_i \prec t_j$ for each simplification order $\succ$ by definition. Hence, we get $t_j \not\succ t_i$ by asymmetry of simplification orders.

Exercise 3 (Termination Proofs with Simplification Orders): (1 + 3 + 2 = 6 points)

Please prove termination of the following TRSs using the embedding order. If this is not possible, use LPO or LPOS instead and explicitly state the precedence (and the status) you are using. In this exercise, x, y, xs, ys, z, and zs denote variables while all other identifiers denote function symbols.

To prove that for two terms t_1 and t_2 we have $t_1 \succ_{emb} t_2$, $t_1 \succ_{lpo} t_2$, or $t_1 \succ_{lpos} t_2$, use a proof tree notation to indicate which case of the definition of $\succ_{emb}$, $\succ_{lpo}$, or $\succ_{lpos}$ you are using. This is illustrated by the following example where we have $t_1 = \mathsf{f}(\mathcal{O}, \mathsf{s}(x))$, $t_2 = \mathsf{f}(\mathsf{s}(\mathcal{O}), x)$, and $t_1 \succ_{lpos} t_2$:
Choose $\mathsf{f} \sqsupset \mathsf{s}$ and $\langle 2, 1 \rangle$ for f. Then we have

$$\dfrac{\dfrac{\overline{x \succeq_{lpos} x}\ {=}}{\mathsf{s}(x) \succ_{lpos} x}\ 1 \qquad \dfrac{\dfrac{\overline{\mathcal{O} \succeq_{lpos} \mathcal{O}}\ {=}}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathcal{O}}\ 1}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathsf{s}(\mathcal{O})}\ 2}{\mathsf{f}(\mathcal{O}, \mathsf{s}(x)) \succ_{lpos} \mathsf{f}(\mathsf{s}(\mathcal{O}), x)}\ 3$$

Hint: You may abbreviate names of function symbols (e.g., "a" instead of "append").

a)

$$\begin{array}{rcl} \mathsf{nth}(\mathcal{O}, \mathsf{Cons}(y, ys)) & \to & y \\ \mathsf{nth}(\mathsf{s}(x), \mathsf{Cons}(y, ys)) & \to & \mathsf{nth}(x, ys) \end{array}$$

b)

$$\begin{array}{rcl} \mathsf{append}(\mathsf{Nil}, ys) & \to & ys \\ \mathsf{append}(\mathsf{Cons}(x, xs), ys) & \to & \mathsf{Cons}(x, \mathsf{append}(xs, ys)) \\ \mathsf{reverse}(\mathsf{Nil}) & \to & \mathsf{Nil} \\ \mathsf{reverse}(\mathsf{Cons}(x, xs)) & \to & \mathsf{append}(\mathsf{reverse}(xs), \mathsf{Cons}(x, \mathsf{Nil})) \end{array}$$

c)

$$\begin{array}{rcl} \mathsf{sum}(x, \mathcal{O}, \mathsf{Nil}) & \to & x \\ \mathsf{sum}(x, \mathsf{s}(y), zs) & \to & \mathsf{sum}(\mathsf{s}(x), y, zs) \\ \mathsf{sum}(x, \mathcal{O}, \mathsf{Cons}(z, zs)) & \to & \mathsf{sum}(x, z, zs) \end{array}$$

Solution:

a)

$$\dfrac{\dfrac{\dfrac{}{y \succeq_{emb} y}\;=}{\mathsf{Cons}(y, ys) \succeq_{emb} y}\;1}{\mathsf{nth}(\mathcal{O}, \mathsf{Cons}(y, ys)) \succ_{emb} y}\;1 \qquad \dfrac{\dfrac{\dfrac{}{x \succeq_{emb} x}\;=}{\mathsf{s}(x) \succ_{emb} x}\;1 \qquad \dfrac{\dfrac{}{ys \succeq_{emb} ys}\;=}{\mathsf{Cons}(y, ys) \succeq_{emb} ys}\;1}{\mathsf{nth}(\mathsf{s}(x), \mathsf{Cons}(y, ys)) \succ_{emb} \mathsf{nth}(x, ys)}\;2$$

b) We choose $\mathsf{r} \sqsupset \mathsf{a} \sqsupset \mathsf{Cons} \sqsupset \mathsf{Nil}$.

$$\dfrac{\dfrac{}{ys \succeq_{lpo} ys}\;=}{\mathsf{a}(\mathsf{Nil}, ys) \succ_{lpo} ys}\;1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\;=}{\mathsf{Cons}(x, xs) \succeq_{lpo} x}\;1}{\mathsf{a}(\mathsf{Cons}(x, xs), ys) \succ_{lpo} x}\;1 \qquad \dfrac{\dfrac{\dfrac{}{xs \succeq_{lpo} xs}\;=}{\mathsf{Cons}(x, xs) \succ_{lpo} xs}\;1 \qquad \dfrac{\dfrac{}{ys \succeq_{lpo} ys}\;=}{\mathsf{a}(\mathsf{Cons}(x, xs), ys) \succ_{lpo} ys}\;1}{\mathsf{a}(\mathsf{Cons}(x, xs), ys) \succ_{lpo} \mathsf{a}(xs, ys)}\;3}{\mathsf{a}(\mathsf{Cons}(x, xs), ys) \succ_{lpo} \mathsf{Cons}(x, \mathsf{a}(xs, ys))}\;2$$

$$\dfrac{\dfrac{}{\mathsf{Nil} \succeq_{lpo} \mathsf{Nil}}\;=}{\mathsf{r}(\mathsf{Nil}) \succ_{lpo} \mathsf{Nil}}\;1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{xs \succeq_{lpo} xs}\;=}{\mathsf{Cons}(x, xs) \succ_{lpo} xs}\;1}{\mathsf{r}(\mathsf{Cons}(x, xs)) \succ_{lpo} \mathsf{r}(xs)}\;3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\;=}{\mathsf{Cons}(x, xs) \succeq_{lpo} x}\;1}{\mathsf{r}(\mathsf{Cons}(x, xs)) \succ_{lpo} x}\;1 \qquad \dfrac{}{\mathsf{r}(\mathsf{Cons}(x, xs)) \succ_{lpo} \mathsf{Nil}}\;2}{\mathsf{r}(\mathsf{Cons}(x, xs)) \succ_{lpo} \mathsf{Cons}(x, \mathsf{Nil})}\;2}{\mathsf{r}(\mathsf{Cons}(x, xs)) \succ_{lpo} \mathsf{a}(\mathsf{r}(xs), \mathsf{Cons}(x, \mathsf{Nil}))}\;2$$

c) We choose $\mathsf{sum} \sqsupset \mathsf{s}$ and $\langle 3, 2, 1 \rangle$ for sum.

$$\dfrac{\dfrac{}{x \succeq_{lpos} x}\;=}{\mathsf{sum}(x, \mathcal{O}, \mathsf{Nil}) \succ_{lpos} x}\;1$$

$$\dfrac{\dfrac{\dfrac{}{y \succeq_{lpos} y}\;=}{\mathsf{s}(y) \succ_{lpos} y}\;1 \qquad \dfrac{\dfrac{\dfrac{}{x \succeq_{lpos} x}\;=}{\mathsf{sum}(x, \mathsf{s}(y), zs) \succ_{lpos} x}\;1}{\mathsf{sum}(x, \mathsf{s}(y), zs) \succ_{lpos} \mathsf{s}(x)}\;2}{\mathsf{sum}(x, \mathsf{s}(y), zs) \succ_{lpos} \mathsf{sum}(\mathsf{s}(x), y, zs)}\;3$$

$$
\cfrac{
\cfrac{\cfrac{}{zs \succeq_{lpos} zs}\,{=}}{\mathsf{Cons}(z, zs) \succ_{lpos} zs}\,1
\qquad
\cfrac{\cfrac{\cfrac{}{z \succeq_{lpos} z}\,{=}}{\mathsf{Cons}(z, zs) \succeq_{lpos} z}\,1}{\mathsf{sum}(x, \mathcal{O}, \mathsf{Cons}(z, zs)) \succeq_{lpos} z}\,1
\qquad
\cfrac{\cfrac{}{x \succeq_{lpos} x}\,{=}}{\mathsf{sum}(x, \mathcal{O}, \mathsf{Cons}(z, zs)) \succ_{lpos} x}\,1
}{\mathsf{sum}(x, \mathcal{O}, \mathsf{Cons}(z, zs)) \succ_{lpos} \mathsf{sum}(x, z, zs)}\,3
$$