Size-Change Termination for Term Rewriting*

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Abstract. In [13], a new *size-change principle* was proposed to verify termination of functional programs automatically. We extend this principle in order to prove termination and innermost termination of arbitrary term rewrite systems (TRSs). Moreover, we compare this approach with existing techniques for termination analysis of TRSs (such as recursive path orderings or dependency pairs). It turns out that the size-change principle on its own fails for many examples that can be handled by standard techniques for rewriting, but there are also TRSs where it succeeds whereas existing rewriting techniques fail. In order to benefit from their respective advantages, we show how to combine the size-change principle with classical orderings and with dependency pairs. In this way, we obtain a new approach for automated termination proofs of TRSs which is more powerful than previous approaches.

1 Introduction

The size-change principle [13] is a new technique for automated termination analysis of functional programs, which raised great interest in the functional programming and automated reasoning community. However, up to now the connection between this principle and existing approaches for termination proofs of term rewriting was unclear. After introducing the size-change principle in Sect. 2, we show how to use it for (innermost) termination proofs of arbitrary TRSs in Sect. 3. This also illustrates how to combine the size-change principle with existing orderings from term rewriting. In Sect. 4 and 5 we compare the size-change principle with classical simplification orderings and with the dependency pair approach [1] for termination of TRSs. Finally, to combine their advantages, we developed a technique which integrates the size-change principle and dependency pairs. The combined technique was implemented in the system AProVE resulting in a very efficient and powerful automated method (details can be found in [14]).

2 The Size-Change Principle

We assume familiarity with term rewriting [3]. For a TRS $\mathcal{R}$ over a signature $\mathcal{F}$, the *defined symbols* $\mathcal{D}$ are the root symbols of the left-hand sides of rules and the *constructors* are $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$. We restrict ourselves to finite signatures and TRSs. $\mathcal{R}$ is a *constructor system* if the left-hand sides of its rules have the form $f(s_1, \ldots, s_n)$ where s_i are *constructor terms* (i.e., $s_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})$). For a signature $\mathcal{F}$, the *embedding rules* $Emb_{\mathcal{F}}$ are $\{f(x_1, \ldots, x_n) \to x_i \mid f \in \mathcal{F}, n = \text{arity}(f), 1 \leq i \leq n\}$.

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In [13], the size-change principle was formulated for a functional programming language with eager evaluation strategy and without pattern matching. Such functional programs are easily transformed into TRSs which are orthogonal constructor systems whose ground normal forms only contain constructors (i.e., all functions are "completely" defined). In this section we present an extension of the original size-change principle which can be used for arbitrary TRSs.

We call $(\succsim, \succ)$ a *reduction pair* [11] on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ if $\succsim$ is a quasi-ordering and $\succ$ is a well-founded ordering where $\succsim$ and $\succ$ are closed under substitutions and compatible (i.e., $\succsim \circ \succ \subseteq \succ$ or $\succ \circ \succsim \subseteq \succ$, but $\succ \subseteq \succsim$ is not required). In general, neither $\succsim$ nor $\succ$ must be closed under contexts. The reduction pair is *monotonic* if $\succsim$ is closed under contexts. In Sect. 3 we examine which additional conditions must be imposed on $(\succsim, \succ)$ in order to use the size-change principle for (innermost) termination proofs of TRSs. *Size-change graphs* denote how the size of function parameters changes when going from one function call to another.

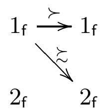
Definition 1 (Size-Change Graph). *Let $(\succsim, \succ)$ be a reduction pair. For every rule $f(s_1, \ldots, s_n) \to r$ of a TRS $\mathcal{R}$ and every subterm $g(t_1, \ldots, t_m)$ of r where $g \in \mathcal{D}$, we define a* size-change graph*. The graph has n* output nodes *marked with $\{1_f, \ldots, n_f\}$ and m* input nodes *marked with $\{1_g, \ldots, m_g\}$. If $s_i \succ t_j$, then there is a directed edge marked with "$\succ$" from output node i_f to input node j_g. Otherwise, if $s_i \succsim t_j$, then there is an edge marked with "$\succsim$" from i_f to j_g. If f and g are clear from the context, then we often omit the subscripts from the nodes. So a size-change graph is a bipartite graph $G = (V, W, E)$ where $V = \{1_f, \ldots, n_f\}$ and $W = \{1_g, \ldots, m_g\}$ are the labels of the output and input nodes, respectively, and we have edges $E \subseteq V \times W \times \{\succsim, \succ\}$.*

Example 2. Let $\mathcal{R}$ consist of the following rules.

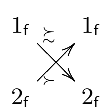
$$\mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(x, \mathsf{s}(x)) \qquad (1) \qquad\qquad \mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(y, x) \qquad (2)$$

$\mathcal{R}$ has two size-change graphs $G_{(1)}$ and $G_{(2)}$ resulting from (1) and (2). Here, we use the embedding ordering on constructors $\mathcal{C}$, i.e., $(\succsim, \succ) = (\to^*_{Emb_\mathcal{C}}, \to^+_{Emb_\mathcal{C}})$.

$G_{(1)}$:



$G_{(2)}$:



To trace sizes of parameters along subsequent function calls, size-change graphs (V_1, W_1, E_1) and (V_2, W_2, E_2) can be concatenated to *multigraphs* if $W_1 = V_2$, i.e., if they correspond to arguments $\{1_g, \ldots, m_g\}$ of the same function g.

Definition 3 (Multigraph and Concatenation). *Every size-change graph of $\mathcal{R}$ is a* multigraph *of $\mathcal{R}$ and if $G = (\{1_f, \ldots, n_f\}, \{1_g, \ldots, m_g\}, E_1)$ and $H = (\{1_g, \ldots, m_g\}, \{1_h, \ldots, p_h\}, E_2)$ are multigraphs w.r.t. the same reduction pair $(\succsim, \succ)$, then the* concatenation *$G \bullet H = (\{1_f, \ldots, n_f\}, \{1_h, \ldots, p_h\}, E)$ is also a multigraph of $\mathcal{R}$. For $1 \le i \le n$ and $1 \le k \le p$, E contains an edge from i_f to k_h iff E_1 contains an edge from i_f to some j_g and E_2 contains an edge from j_g to k_h. If there is such a j_g where the edge of E_1 or E_2 is labelled with "$\succ$", then the edge in E is labelled with "$\succ$" as well. Otherwise, it is labelled with "$\succsim$". A multigraph G is called* maximal *if its input and output nodes are both* *labelled with* $\{1_f, \ldots, n_f\}$ *for some* f *and if it is idempotent, i.e.,* $G = G \bullet G$.

Example 4. In Ex. 2 we obtain the following three maximal multigraphs:

$$G_{(1)} \bullet G_{(2)} : \begin{tikzcd} 1_{\mathsf{f}} \arrow[r, "\succ"] \arrow[dr, "\succ"] & 1_{\mathsf{f}} \\ 2_{\mathsf{f}} & 2_{\mathsf{f}} \end{tikzcd} \qquad G_{(2)} \bullet G_{(1)} : \begin{tikzcd} 1_{\mathsf{f}} & 1_{\mathsf{f}} \\ 2_{\mathsf{f}} \arrow[ur, "\succ"] \arrow[r, "\succ"'] & 2_{\mathsf{f}} \end{tikzcd} \qquad G_{(2)} \bullet G_{(2)} : \begin{tikzcd} 1_{\mathsf{f}} \arrow[r, "\succ"] & 1_{\mathsf{f}} \\ 2_{\mathsf{f}} \arrow[r, "\succ"] & 2_{\mathsf{f}} \end{tikzcd}$$

For termination, in every maximal multigraph a parameter must be decreasing.[1]

Definition 5 (Size-Change Termination). *A TRS* $\mathcal{R}$ *over the signature* $\mathcal{F}$ *is* size-change terminating *w.r.t. a reduction pair* $(\succsim, \succ)$ *on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ *iff every maximal multigraph contains an edge of the form* $i \xrightarrow{\succ} i$.

In Ex. 4, each maximal multigraph contains the edge $1_{\mathsf{f}} \xrightarrow{\succ} 1_{\mathsf{f}}$ or $2_{\mathsf{f}} \xrightarrow{\succ} 2_{\mathsf{f}}$. So the TRS is size-change terminating w.r.t. the embedding ordering. Note that classical path orderings from term rewriting fail on this example (see Sect. 4).

Since there are only finitely many possible multigraphs, they can be constructed automatically. So for a given reduction pair, size-change termination is decidable. However, in general size-change termination does not imply termination.

Example 6. Consider the TRS with the rules $\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b})$ and $\mathsf{b} \to \mathsf{a}$. If we use the lexicographic path ordering $\succ_{LPO}$ [9] with the precedence $\mathsf{a} > \mathsf{b}$, then the only maximal multigraph is $1_{\mathsf{f}} \xrightarrow{\succ_{LPO}} 1_{\mathsf{f}}$. So size-change termination is proved, although the TRS is obviously not terminating.

3 Size-Change Termination and Termination of TRSs

In this section we develop conditions on the reduction pair used in Def. 5 which ensure that size-change termination indeed implies (innermost) termination. Then the size-change principle can be combined with classical orderings from term rewriting and it becomes a sound termination criterion.

In [13], the authors use reduction pairs $(\succsim, \succ)$ where $\succsim$ is the reflexive closure of $\succ$ and $\succ$ is defined in terms of a well-founded relation $>$ on (ground) normal forms of $\mathcal{R}$. We now show that such reduction pairs can be used for innermost termination proofs of arbitrary TRSs. Moreover, $\succsim$ can be any compatible quasi-ordering. We denote innermost reduction steps by $\xrightarrow{\mathrm{i}}_{\mathcal{R}}$ and $s \xrightarrow{\mathrm{i}}{}^{!}_{\mathcal{R}} s'$ means that s' is a normal form reachable from s by innermost reduction. Thm. 7 will serve as the basis for the automation of the size-change principle in Thm. 8 afterwards.

Theorem 7 (Size-Change Termination and Innermost Termination). *Let* $>$ *be a well-founded ordering on normal forms of a TRS* $\mathcal{R}$. *For* $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ *we define* $\mathrm{NF}(s,t) = \{(s', t') \mid s\sigma \xrightarrow{\mathrm{i}}{}^{!}_{\mathcal{R}} s', t\sigma \xrightarrow{\mathrm{i}}{}^{!}_{\mathcal{R}} t', \sigma$ *instantiates all variables of* s *and* t *with normal forms of* $\mathcal{R}\}$. *Let* $(\succsim, \succ)$ *be a reduction pair where* $s \succ t$ *implies* $s' > t'$ *for all* $(s', t') \in \mathrm{NF}(s,t)$. *If* $\mathcal{R}$ *is size-change terminating w.r.t.* $(\succsim, \succ)$, *then* $\mathcal{R}$ *is innermost terminating.*

Proof. If $\mathcal{R}$ is not innermost terminating, then there is a minimal non-innermost

[1] Def. 5 corresponds to an equivalent characterization of size-change termination [13].

terminating term v_0, i.e., all proper subterms of v_0 are innermost terminating. Let $\xrightarrow{\mathrm{i}}_\epsilon$ denote root reductions and let $\xrightarrow{\mathrm{i}}_{>\epsilon}$ denote reductions below the root. Then v_0's infinite innermost reduction starts with $v_0 \xrightarrow{\mathrm{i}}^*_{>\epsilon} u_1 \xrightarrow{\mathrm{i}}_\epsilon w_1$ where all proper subterms of u_1 are in normal form. Since w_1 is not innermost terminating, it has a minimal non-innermost terminating subterm v_1.

The infinite reduction continues in the same way. So for $i \geq 1$, we have $v_{i-1} \xrightarrow{\mathrm{i}}^*_{>\epsilon} u_i = l_i\sigma_i$ and $v_i = r'_i\sigma_i$ for a rule $l_i \to r_i$, a subterm r'_i of r_i with defined root, and a substitution σ_i instantiating l_i's variables with normal forms.

For each step from u_i to v_i there is a corresponding size-change graph G_i. We regard the infinite graph resulting from $G_1, G_2, \ldots$ by identifying the input nodes of G_i with the output nodes of G_{i+1}. If $\mathcal{R}$ is size-change terminating, by [13, Thm. 4] resp. [14, Lemma 7], this infinite graph contains an infinite path where infinitely many edges are labelled with "$\succ$". Without loss of generality we assume that this path already starts in G_1. For every i, let a_i be the output node in G_i which is on this path. So we have $l_i|_{a_i} \succ r'_i|_{a_{i+1}}$ for all i from an infinite set $I \subseteq \mathbb{N}$ and $l_i|_{a_i} \succsim r'_i|_{a_{i+1}}$ for $i \in \mathbb{N}\backslash I$. Note that $l_i|_{a_i}\sigma_i = u_i|_{a_i}$ and $r'_i|_{a_{i+1}}\sigma_i = v_i|_{a_{i+1}} \xrightarrow{\mathrm{i}}^!_{\mathcal{R}} u_{i+1}|_{a_{i+1}}$. Thus, $(u_i|_{a_i}, u_{i+1}|_{a_{i+1}}) \in \mathrm{NF}(l_i|_{a_i}, r'_i|_{a_{i+1}})$. So for $I = \{i_1, i_2, \ldots\}$ we obtain $u_{i_1}|_{a_{i_1}} > u_{i_2}|_{a_{i_2}} > \ldots$ which contradicts well-foundedness of $>$. $\square$

Innermost termination is interesting, since then there are no infinite reductions w.r.t. eager evaluation strategies. Moreover, for non-overlapping TRSs, innermost termination already implies termination. However, Thm. 7 is not yet suitable for automation. To check whether $\succ$ satisfies the conditions of Thm. 7, one has to examine infinitely many instantiations of s and t and compute normal forms s' and t' although $\mathcal{R}$ is possibly not innermost terminating. Therefore, in the examples of [13], one is restricted to relations $\succsim$ and $\succ$ on constructor terms.

Thm. 8 shows how to use reduction pairs on $\mathcal{T}(\mathcal{C}, \mathcal{V})$ for possibly automated innermost termination proofs. A reduction pair $(\succsim, \succ)$ on $\mathcal{T}(\mathcal{G}, \mathcal{V})$ with $\mathcal{G} \subseteq \mathcal{F}$ can be *extended* to a (usually non-monotonic) reduction pair $(\succsim', \succ')$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ by defining $s \succsim' t$ if $s = t$ or if there are $u, v \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ with $u \succsim v$ and $s = u\sigma$, $t = v\sigma$ for some substitution σ. Moreover, $s \succ' t$ iff $u \succ v$ for u and v as above.

Theorem 8 (Innermost Termination Proofs). *Let $(\succsim, \succ)$ be a reduction pair on $\mathcal{T}(\mathcal{C}, \mathcal{V})$. If $\mathcal{R}$ is size-change terminating w.r.t. the extension of the reduction pair $(\succsim, \succ)$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, then $\mathcal{R}$ is innermost terminating.*

Proof. Let $(\succsim', \succ')$ be the extension of $(\succsim, \succ)$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We show that $s \succ' t$ implies $s' \succ' t'$ for all $(s', t') \in \mathrm{NF}(s, t)$. Then the theorem follows from Thm. 7.

By the definition of extensions, $s \succ' t$ iff $s = u\sigma$, $t = v\sigma$, and $u \succ v$ for suitable u, v, and σ. In particular, u and v must be constructor terms and we also have $u \succ' v$ (as σ may also be the identity). Since $\mathrm{NF}(s, t) \subseteq \{(u\sigma, v\sigma) \mid \sigma$ instantiates u's and v's variables with normal forms$\}$, the claim follows from $u \succ' v$, because $\succ'$ is closed under substitutions. $\square$

For the TRS in Ex. 2, when using the extension of the reduction pair $(\to^*_{Emb_\mathcal{C}}, \to^+_{Emb_\mathcal{C}})$ on $\mathcal{T}(\mathcal{C}, \mathcal{V})$, we obtain the same size-change graphs as with $(\to^*_{Emb_\mathcal{C}}, \to^+_{Emb_\mathcal{C}})$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Ex. 4 shows that this TRS is size-change terminating w.r.t. this reduction pair and hence, by Thm. 8, this proves innermost termination. However, a variant of Toyama's example [15] shows that Thm. 7 and Thm. 8 are not sufficient to prove full (non-innermost) termination.

Example 9. Let $\mathcal{R} = \{\mathsf{f}(\mathsf{c}(\mathsf{a},\mathsf{b},x)) \to \mathsf{f}(\mathsf{c}(x,x,x)),\ \mathsf{g}(x,y) \to x,\ \mathsf{g}(x,y) \to y\}$. We define $\succsim = \to_{\mathcal{S}}^*$ and $\succ = \to_{\mathcal{S}}^+$ restricted to $\mathcal{T}(\mathcal{C},\mathcal{V})$, where $\mathcal{S}$ is the terminating TRS $\mathsf{c}(\mathsf{a},\mathsf{b},x) \to \mathsf{c}(x,x,x)$. The only maximal multigraph is $1_\mathsf{f} \xrightarrow{\succsim} 1_\mathsf{f}$. So $\mathcal{R}$ is size-change terminating and innermost terminating by Thm. 8, but not terminating.

As in Ex. 9, reduction pairs $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$ satisfying the conditions of Thm. 8 can be defined using a terminating TRS $\mathcal{S}$ over the signature $\mathcal{C}$. The following theorem shows that if $\mathcal{S}$ is non-duplicating, then we may use the relation $\to_{\mathcal{S}}$ also on terms with defined symbols and size-change termination even implies full termination. A TRS is *non-duplicating* if every variable occurs on the right-hand side of a rule at most as often as on the corresponding left-hand side. So size-change termination of the TRS in Ex. 2 and Ex. 4 using the reduction pair $(\to_{Emb_\mathcal{C}}^*, \to_{Emb_\mathcal{C}}^+)$ implies that the TRS is indeed terminating.

To prove the theorem, we need a preliminary lemma which states that minimal non-terminating terms w.r.t. $\mathcal{R} \cup \mathcal{S}$ cannot start with constructors of $\mathcal{R}$. Again, here $\mathcal{S}$ must be non-duplicating. Otherwise, in Ex. 9, $\mathsf{c}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b}))$ is a minimal non-terminating term w.r.t. $\mathcal{R} \cup \mathcal{S}$ that starts with a constructor of $\mathcal{R}$.

Lemma 10. *Let $\mathcal{R}$ be a TRS over the signature $\mathcal{F}$ with constructors $\mathcal{C}$ and let $\mathcal{S}$ be a terminating non-duplicating TRS over $\mathcal{C}$. If $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ are terminating w.r.t. $\mathcal{R} \cup \mathcal{S}$ and $c \in \mathcal{C}$, then $c(t_1,\ldots,t_n)$ is also terminating w.r.t. $\mathcal{R} \cup \mathcal{S}$.*

Proof. For any term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$, let M_s be the multiset of the maximal subterms of s whose root is defined, i.e., $M_s = \{s|_\pi \mid root(s|_\pi) \in \mathcal{D}$ and for all π' above π we have $root(s|_{\pi'}) \in \mathcal{C}\}$. Moreover, let s' be the term that results from s by replacing all maximal subterms with defined root by the same fresh special variable $x_\mathcal{C}$. Let $\twoheadrightarrow_{\mathcal{R}\cup\mathcal{S}}$ be the extension of $\to_{\mathcal{R}\cup\mathcal{S}}$ to multisets where $M \twoheadrightarrow_{\mathcal{R}\cup\mathcal{S}} M'$ iff $M = N \cup \{s\}$ and $M' = N \cup \{t_1,\ldots,t_n\}$ with $n \geq 0$ and with $s \to_{\mathcal{R}\cup\mathcal{S}} t_i$ for all i. We prove the following conjecture.

> Let $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ such that all terms in M_s are terminating w.r.t. $\mathcal{R} \cup \mathcal{S}$ and let $s \to_{\mathcal{R}\cup\mathcal{S}} t$. Then all terms in M_t are also terminating w.r.t. $\mathcal{R}\cup\mathcal{S}$. Moreover, $M_s \twoheadrightarrow_{\mathcal{R}\cup\mathcal{S}} M_t$ or both $M_s = M_t$ and $s' \to_\mathcal{S} t'$. (3)

Note that $\twoheadrightarrow_{\mathcal{R}\cup\mathcal{S}}$ is well founded on multisets like M_s which only contain terminating terms. Termination of $\mathcal{S}$ implies that $\to_\mathcal{S}$ is also well founded and the lexicographic combination of two well-founded orderings preserves well-foundedness. Hence, (3) implies that if all terms in M_s are terminating, then s is terminating as well. So the lemma immediately follows from Conjecture (3).

To prove (3), we distinguish according to the position π where the reduction $s \to_{\mathcal{R}\cup\mathcal{S}} t$ takes place. If s has a defined symbol of $\mathcal{D}$ on or above position π, then this implies $M_s \twoheadrightarrow_{\mathcal{R}\cup\mathcal{S}} M_t$ and all terms in M_t are also terminating. Otherwise, if π is above all symbols of $\mathcal{D}$ in s, then $s \to_{\mathcal{R}\cup\mathcal{S}} t$ implies $s \to_\mathcal{S} t$ and $M_s \supseteq M_t$ (since $\mathcal{S}$ is non-duplicating). Moreover, $s \to_\mathcal{S} t$ also implies $s' \to_\mathcal{S} t'$. ☐

Theorem 11 (Termination Proofs). *Let $\mathcal{R}$ be a TRS over the signature $\mathcal{F}$ with constructors $\mathcal{C}$ and let $\mathcal{S}$ be a terminating non-duplicating TRS over $\mathcal{C}$. If $\mathcal{R}$*

is size-change terminating w.r.t. the reduction pair $(\to_{\mathcal{S}}^{*}, \to_{\mathcal{S}}^{+})$ *on* $\mathcal{T}(\mathcal{F},\mathcal{V})$*, then* $\mathcal{R}$ *(and even* $\mathcal{R} \cup \mathcal{S}$*) is terminating.*

Proof. We define $\mathcal{R}' := \mathcal{R} \cup \mathcal{S}$. If $\mathcal{R}'$ is not terminating, then as in the proof of Thm. 7 we obtain an infinite sequence of minimal non-terminating terms u_i, v_i with $v_i \to_{>\epsilon,\mathcal{R}'}^{*} u_{i+1}$ where the step from u_i to v_i corresponds to a size-change graph of $\mathcal{R}'$. Thus, for all i there is a rule $l_i \to r_i$ in $\mathcal{R}'$ with $u_i = l_i\sigma_i$ and $v_i = r_i'\sigma_i$ for a subterm r_i' of r_i and a substitution σ_i.

By Lemma 10, the roots of u_i and v_i are defined symbols. So all these size-change graphs are from $\mathcal{R}$. As in Thm. 7's proof, there are a_i with $l_i|_{a_i} \to_{\mathcal{S}}^{+} r_i'|_{a_{i+1}}$ for all i from an infinite $I \subseteq \mathbb{N}$ and $l_i|_{a_i} \to_{\mathcal{S}}^{*} r_i'|_{a_{i+1}}$ for $i \in \mathbb{N} \setminus I$. Since $\to_{\mathcal{S}}$ is closed under substitution we have $u_i|_{a_i} \to_{\mathcal{S}}^{+} v_i|_{a_{i+1}}$ or $u_i|_{a_i} \to_{\mathcal{S}}^{*} v_i|_{a_{i+1}}$, respectively. Recall $v_i|_{a_{i+1}} \to_{\mathcal{R}'}^{*} u_{i+1}|_{a_{i+1}}$ and $\mathcal{S} \subseteq \mathcal{R}'$. So for $I = \{i_1, i_2, \ldots\}$ we have $u_{i_1}|_{a_{i_1}} \to_{\mathcal{R}'}^{+} u_{i_2}|_{a_{i_2}} \to_{\mathcal{R}'}^{+} \ldots$ contradicting the minimality of the terms u_i. □

Thm. 8 and 11 offer two possibilities for automating the size-change principle. Even for innermost termination, they do not subsume each other. Ex. 9 cannot be handled by Thm. 11 and innermost termination of $\{\mathsf{g}(\mathsf{f}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{b})), \mathsf{f}(x) \to x\}$ cannot be proved with Thm. 8, since $\mathsf{f}(\mathsf{a}) \not\succ \mathsf{f}(\mathsf{b})$ for any extension $\succ$ of an ordering on constructor terms. But termination is shown with Thm. 11 using $\mathcal{S} = \{\mathsf{a} \to \mathsf{b}\}$. A variant of Thm. 11 for innermost termination holds if $\mathcal{S}$ is innermost terminating (and possibly duplicating). However, this variant only proves innermost termination of $\mathcal{R} \cup \mathcal{S}$ which does not imply innermost termination of $\mathcal{R}$.

So Thm. 8 and Thm. 11 are new contributions that show which reduction pairs are admissible in order to use size-change termination for termination or innermost termination proofs of TRSs. In this way, size-change termination becomes an automatic technique, since one can use classical techniques from termination of term rewriting to generate suitable reduction pairs automatically.

4 Comparison with Orderings from Term Rewriting

Traditional techniques for TRSs prove *simple termination* where $\mathcal{R}$ is simply terminating iff it is compatible with a *simplification ordering* (e.g., LPO or RPOS [5, 9], KBO [10], most polynomial orderings [12]). Equivalently, $\mathcal{R}$ is simply terminating iff $\mathcal{R} \cup \mathit{Emb}_{\mathcal{F}}$ terminates for $\mathcal{R}$'s signature $\mathcal{F}$. Similar to traditional techniques, the size-change principle essentially only verifies simple termination.

Theorem 12 (Size-Change Principle and Simple Termination).

(a) A TRS $\mathcal{R}$ *over a signature* $\mathcal{F}$ *is size-change terminating w.r.t. a reduction pair* $(\succsim, \succ)$ *iff* $\mathcal{R} \cup \mathit{Emb}_{\mathcal{F}}$ *is size-change terminating w.r.t.* $(\succsim, \succ)$*.*
(b) Let $\mathcal{S}$ *be as in Thm. 11. If* $\mathcal{S}$ *is simply terminating and* $\mathcal{R}$ *is size-change terminating w.r.t.* $(\to_{\mathcal{S}}^{*}, \to_{\mathcal{S}}^{+})$ *on* $\mathcal{T}(\mathcal{F},\mathcal{V})$*, then* $\mathcal{R} \cup \mathcal{S}$ *is simply terminating.*

Proof. (a) The "if" direction is obvious. For the "only if" direction, note that $\mathit{Emb}_{\mathcal{F}}$ yields no new size-change graphs. But due to $\mathit{Emb}_{\mathcal{C}}$, all constructors are transformed into defined symbols. So from the $\mathcal{R}$-rules we obtain additional size-change graphs whose input nodes are labelled with (former) constructors (i.e., $1_c, \ldots, n_c$ for $c \in \mathcal{C}$). However, since output nodes are never labelled with constructors, this does not yield new maximal multigraphs (since there, output and input nodes are labelled by the same function). Hence, size-change termination is not affected when adding $Emb_{\mathcal{F}}$.

(b) As in (a), adding $Emb_{\mathcal{D}}$ to $\mathcal{R}$ yields no new size-change graphs and thus, $\mathcal{R} \cup Emb_{\mathcal{D}}$ is also size-change terminating w.r.t. $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$ and hence, also w.r.t. $(\to_{\mathcal{S} \cup Emb_{\mathcal{C}}}^*, \to_{\mathcal{S} \cup Emb_{\mathcal{C}}}^+)$. Since $S \cup Emb_{\mathcal{C}}$ is terminating, Thm. 11 implies termination of $\mathcal{R} \cup Emb_{\mathcal{D}} \cup \mathcal{S} \cup Emb_{\mathcal{C}}$, i.e., simple termination of $\mathcal{R} \cup \mathcal{S}$. $\square$

The restriction to simple termination excludes many relevant TRSs. Thm. 12 illustrates that the size-change principle cannot compete with new techniques (e.g., *dependency pairs* [1] or *monotonic semantic path ordering* [4]) where simplification orderings may be applied to non-simply terminating TRSs as well. However, these new techniques require methods to generate underlying base orderings. Hence, there is still an urgent need for powerful simplification orderings.

Now we clarify the connection between size-change termination and classical simplification orderings and show that they do not subsume each other in general.

A major advantage of the size-change principle is that it can simulate the basic ingredients of RPOS, i.e., the concepts of *lexicographic* and of *multiset*-comparison. Thus, by the size-change principle w.r.t. a very simple reduction pair like the embedding ordering we obtain an automated method for termination analysis which avoids the search problems of RPOS and which can still capture the idea of comparing tuples of arguments lexicographically or as multisets.

More precisely, for a reduction pair $(\succsim, \succ)$, let $\succ_{lex}$ and $\succ_{mul}$ result from comparing tuples s^* and t^* of terms lexicographically and as multisets, respectively. If $s_i^* \succ_{lex} t_i^*$ for all $1 \leq i \leq k$, then the TRS $\{f(s_1^*) \to f(t_1^*), \ldots, f(s_k^*) \to f(t_k^*)\}$ is size-change terminating w.r.t. $(\succsim, \succ)$. In particular, size-change termination w.r.t. the same reduction pair $(\succsim, \succ)$ can simulate $\succ_{lex}$ for *any permutation* used to compare the components of a tuple. Similarly, if $s_i^* \succ_{mul} t_i^*$ for all i, then this TRS is also size-change terminating w.r.t. $(\succsim, \succ)$.[2] For example, the TRS computing the Ackermann function as well as the TRS $\{\mathsf{plus}(\mathsf{0}, y) \to y,\ \mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{plus}(y, x))\}$ are size-change terminating w.r.t. the embedding ordering on constructors whereas traditional rewriting techniques would need lexicographic and recursive (multiset) path ordering, respectively.

Since both lexicographic and multiset comparison are simulated by the size-change principle using the *same* reduction pair, one can also handle TRSs like Ex. 2 where traditional path orderings like RPOS (or KBO) fail. In the first rule $\mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(x, \mathsf{s}(x))$ the arguments of f have to be compared lexicographically from left to right and in the second rule $\mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(y, x)$ they have to be compared as multisets. If one adds the rules for the Ackermann function then polynomial orderings fail as well, but size-change termination is proved as before.

However, compared to classical path orderings, the size-change principle also has several drawbacks. One problem is that it can only simulate lexicographic and multiset comparison for the arguments of the *root* symbol. Hence, if one adds a new function on top of all terms in the rules, this simulation is no longer possible. For example, the TRS $\{\mathsf{f}(\mathsf{plus}(\mathsf{0}, y)) \to \mathsf{f}(y), \mathsf{f}(\mathsf{plus}(\mathsf{s}(x), y)) \to \mathsf{f}(\mathsf{s}(\mathsf{plus}(y, x)))\}$ is

[2] Formal proofs for these observations can be found in [14, Thm. 14 and Thm. 15].

no longer size-change terminating w.r.t. the embedding ordering, whereas classical path orderings can apply lexicographic or multiset comparisons on all levels of the term. Thus, termination would still be easy to prove with RPO.

Perhaps the most serious drawback is that the size-change principle lacks concepts to compare defined function symbols syntactically. Consider a TRS with the rule $\mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{log}(\mathsf{s}(\mathsf{half}(x))))$ and rules for half such that $\mathsf{half}(x)$ computes $\lfloor \frac{x}{2} \rfloor$. If a function (like log) calls another defined function (like half) in the arguments of its recursive calls, one has to check whether the argument $\mathsf{half}(x)$ is smaller than the term $\mathsf{s}(x)$ in the corresponding left-hand side. The size-change principle on its own offers no possibility for that and its mechanizable versions (Thm. 8 and Thm. 11) fail since they only use an underlying ordering on constructor terms. In contrast, classical orderings like RPO can easily show termination automatically using a *precedence* $\mathsf{log} > \mathsf{s} > \mathsf{half}$ on function symbols.

Finally, the size-change principle has the disadvantage that it cannot *measure* terms by combining measures of subterms as in polynomial orderings or KBO.

Example 13. Measures (*weights*) are especially useful if one parameter is increasing, but the decrease of another parameter is greater than this increase. So termination of $\{\mathsf{plus}(\mathsf{s}(\mathsf{s}(x)), y) \to \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(y))),\ \mathsf{plus}(x, \mathsf{s}(\mathsf{s}(y))) \to \mathsf{s}(\mathsf{plus}(\mathsf{s}(x), y)),$ $\mathsf{plus}(\mathsf{s}(0), y) \to \mathsf{s}(y),\ \mathsf{plus}(0, y) \to y\}$ is trivial to prove with polynomial orderings or KBO, but the TRS is not size-change terminating w.r.t. *any* reduction pair.

5 Comparison and Combination with Dependency Pairs

Now we compare the size-change principle with *dependency pairs*. In contrast to other recent techniques [4, 6], dependency pairs and size-change graphs are both built from recursive calls which suggests to combine these approaches to benefit from their respective advantages. We recapitulate the concepts of dependency pairs; see [1, 7, 8] for refinements and motivations. Let $\mathcal{F}^\sharp = \{f^\sharp \mid f \in \mathcal{D}\}$ be a set of *tuple symbols*, where $f^\sharp$ has the same arity as f and we often write F for $f^\sharp$, etc. If $t = g(t_1, \ldots, t_m)$ with $g \in \mathcal{D}$, we write $t^\sharp$ for $g^\sharp(t_1, \ldots, t_m)$. If $l \to r \in \mathcal{R}$ and t is a subterm of r with defined root, then the rule $l^\sharp \to t^\sharp$ is a *dependency pair* of $\mathcal{R}$. So the dependency pairs of the TRS from Ex. 2 are

$$\mathsf{F}(\mathsf{s}(x), y) \to \mathsf{F}(x, \mathsf{s}(x)) \qquad (4) \qquad\qquad \mathsf{F}(x, \mathsf{s}(y)) \to \mathsf{F}(y, x) \qquad (5)$$

We always assume that different occurrences of dependency pairs are variable disjoint. Then a TRS is (innermost) terminating iff there is no infinite (innermost) *chain* of dependency pairs. A sequence $s_1 \to t_1, s_2 \to t_2, \ldots$ of dependency pairs is a *chain* iff $t_i\sigma \to_{\mathcal{R}}^* s_{i+1}\sigma$ for all i and a suitable substitution σ. The sequence is an *innermost* chain iff $t_i\sigma \xrightarrow{\mathrm{i}}_{\mathcal{R}}^* s_{i+1}\sigma$ and all $s_i\sigma$ are in normal form.

To estimate which dependency pairs may occur consecutively in chains, one builds a so-called dependency graph. Let $\textsc{cap}(t)$ result from replacing all subterms of t with defined root symbol by different fresh variables and let $\textsc{ren}(t)$ result from replacing all occurrences of variables in t by different fresh variables. For instance, $\textsc{cap}(\mathsf{F}(x, \mathsf{s}(x))) = \mathsf{F}(x, \mathsf{s}(x))$ and $\textsc{ren}(\mathsf{F}(x, \mathsf{s}(x))) = \mathsf{F}(x_1, \mathsf{s}(x_2))$. The (estimated) *dependency graph* is the directed graph whose nodes are the dependency pairs and there is an arc from $s \to t$ to $v \to w$ iff $\textsc{ren}(\textsc{cap}(t))$ and v are unifiable. In the (estimated) *innermost dependency graph* there is only an arc from $s \to t$ to $v \to w$ iff $\textsc{cap}(t)$ and v are unifiable. For the TRS of Ex. 2, the dependency graph and the innermost dependency graph are identical and each dependency pair is connected with itself and with the other pair.

A non-empty set $\mathcal{P}$ of dependency pairs is a *cycle* if for any pairs $s \to t$ and $v \to w$ in $\mathcal{P}$ there is a non-empty path from $s \to t$ to $v \to w$ which only traverses pairs from $\mathcal{P}$. In our example we have the cycles $\{(4)\}$, $\{(5)\}$, and $\{(4),(5)\}$. If a cycle only contains dependency pairs resulting from the rules $\mathcal{R}' \subseteq \mathcal{R}$ we speak of an $\mathcal{R}'$*-cycle* of the dependency graph of $\mathcal{R}$. Finally, for $f \in \mathcal{D}$ we define its *usable rules* $\mathcal{U}(f)$ as the smallest set containing all f-rules and all rules that are usable for function symbols occurring in right-hand sides of f-rules. In our example, the usable rules for f are (1) and (2). For $\mathcal{D}' \subseteq \mathcal{D}$ let $\mathcal{U}(\mathcal{D}') = \bigcup_{f \in \mathcal{D}'} \mathcal{U}(f)$.

Theorem 14 (Dependency Pair Approach [1]). *A TRS $\mathcal{R}$ is terminating iff for each cycle $\mathcal{P}$ in the dependency graph there is a monotonic reduction pair $(\succsim, \succ)$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\sharp, \mathcal{V})$ such that*

(a) $s \succsim t$ for all $s \to t \in \mathcal{P}$ and $s \succ t$ for at least one $s \to t \in \mathcal{P}$
(b) $l \succsim r$ for all $l \to r \in \mathcal{R}$.

$\mathcal{R}$ is innermost terminating if for each cycle $\mathcal{P}$ in the innermost dependency graph there is a monotonic reduction pair $(\succsim, \succ)$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\sharp, \mathcal{V})$ such that

(c) $s \succsim t$ for all $s \to t \in \mathcal{P}$ and $s \succ t$ for at least one $s \to t \in \mathcal{P}$
(d) $l \succsim r$ for all $l \to r \in \mathcal{U}(\mathcal{D}')$,
where $\mathcal{D}' = \{f \mid f \in \mathcal{D}$ occurs in t for some $s \to t \in \mathcal{P}\}$.

For the TRS of Ex. 2, in $\mathcal{P} = \{(4),(5)\}$ we must find a reduction pair where one dependency pair is weakly (w.r.t. $\succsim$) and one is strictly decreasing (w.r.t. $\succ$). Since $\succ$ does not have to be monotonic, one typically uses a standard simplification ordering combined with an *argument filtering* to eliminate argument positions of function symbols. For example, we may eliminate the second argument position of F. Then F becomes unary and every term $\mathsf{F}(s,t)$ is replaced by $\mathsf{F}(s)$. The constraint $\mathsf{F}(\mathsf{s}(x)) \succ \mathsf{F}(x)$ resulting from Dependency Pair (4) is easily satisfied but there is no reduction pair satisfying $\mathsf{F}(x) \succsim \mathsf{F}(y)$ from Dependency Pair (5). Indeed, there is no argument filtering such that the constraints of the dependency pair approach would be satisfied by a standard path ordering like RPOS or KBO. Moreover, if one adds the rules $\mathsf{f}(x,y) \to \mathsf{ack}(x,y)$, $\mathsf{ack}(\mathsf{s}(x),y) \to \mathsf{f}(x,x)$, and the rules for the Ackermann function ack, then the dependency pair constraints are not satisfied by any polynomial ordering either.

Thus, termination cannot be proved with dependency pairs in combination with classical orderings amenable to automation, whereas the proof is very easy with the size-change principle and a simple reduction pair like the embedding ordering on constructors. While the examples in [13] are easily handled by dependency pairs and RPOS, this shows that there exist TRSs where the size-change principle is preferable to dependency pairs and standard rewrite orderings.

In fact, size-change termination encompasses the concept of argument filtering for root symbols, since it concentrates on certain arguments of (root) function symbols while ignoring others. This is an advantage compared to dependency pairs where finding the argument filtering is a major search problem. Moreover, the size-change principle examines sequences of function calls in a more sophisticated way. Depending on the different "paths" from one function call to another, it can choose different arguments to be (strictly) decreasing. In contrast, in the dependency pair approach such choices remain fixed for the whole cycle.

But in addition to the drawbacks in Sect. 4, a disadvantage of the size-change principle is that it is not modular, i.e., one has to use the same reduction pair for the whole termination proof whereas dependency pairs permit different orderings for different cycles. The size-change principle also does not analyze arguments of terms to check whether two function calls can follow each other, whereas in dependency graphs, this is approximated using CAP and REN. Again, the most severe drawback is that the size-change principle offers no technique to compare terms with defined symbols, whereas dependency pairs use inequalities of the form $l \succsim r$ for this purpose. Therefore, only very restricted reduction pairs may be used for the size-change principle in Thm. 8 and 11, whereas one may use arbitrary monotonic reduction pairs for the dependency pair approach. In fact, dependency pairs are a *complete* technique which can prove termination of every TRS, which is not true for the size-change principle (see e.g., Ex. 13).

Therefore, we introduce a new technique to combine dependency pairs and size-change termination. A straightforward approach would be to use size-change termination as the "base ordering" when trying to satisfy the constraints resulting from the dependency pair approach. However, this would be very weak due to the restrictions on the reduction pairs in Thm. 8 and Thm. 11. Instead, we incorporate the size-change principle into the dependency pair approach and use it when generating the constraints. The resulting technique is stronger than both previous approaches: If (innermost) termination can be proved by the size-change principle or by dependency pairs using certain reduction pairs, then it can also be proved with our new technique using the *same* reduction pairs. On the other hand, there are many examples which cannot be proved by the size-change principle and where dependency pairs would require complicated reduction pairs (that can hardly be generated automatically), whereas with our combined technique the (automatic) proof works with very simple reduction pairs, cf. [14].

Obviously, size-change graphs and dependency pairs have a close correspondence, since they both represent a call of a defined symbol g in the right-hand side of a rewrite rule $f(s_1, \ldots, s_n) \to \ldots g(t_1, \ldots, t_m) \ldots$ Since we only need to concatenate size-change graphs which correspond to cycles in the (innermost) dependency graph, we now label size-change graphs by the corresponding dependency pair and multigraphs are labelled by the corresponding sequence of dependency pairs. Then two size-change graphs or multigraphs labelled with $(\ldots, D)$ and $(D', \ldots)$ may only be concatenated if there is an arc from D to D' in the (innermost)[3] dependency graph. Another problem is that in size-change graphs one only has output nodes $1_f, \ldots, n_f$ and input nodes $1_g, \ldots, m_g$ to

[3] Whether one regards the dependency graph or the innermost dependency graph depends on whether one wants to prove termination or innermost termination.

compare the *arguments* of f and g. Therefore, the size-change principle cannot deal with TRSs like Ex. 13 where one has to regard the *whole* term in order to show termination. For that reason we add another output node ϵ_f and input node ϵ_g which correspond to the whole terms (or more precisely, to the terms $F(s_1, \ldots, s_n)$ and $G(t_1, \ldots, t_m)$ of the corresponding dependency pair).

Definition 15 (Extended Size-Change Graphs). *Let $(\succsim, \succ)$ be a reduction pair on $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\sharp, \mathcal{V})$. For every $f(s_1, \ldots, s_n) \to r \in \mathcal{R}$ and subterm $g(t_1, \ldots, t_m)$ of r with $g \in \mathcal{D}$, the* extended size-change graph *has $n+1$ output nodes i_f and $m+1$ input nodes j_g where $i \in \{\epsilon, 1, \ldots, n\}$, $j \in \{\epsilon, 1, \ldots, m\}$. Let $s = F(s_1, \ldots, s_n)$ and $t = G(t_1, \ldots, t_m)$. Then there is an edge $i_f \xrightarrow{\succ} j_g$ iff $s|_i \succ t|_j$ and otherwise, there is an edge $i_f \xrightarrow{\succsim} j_g$ iff $s|_i \succsim t|_j$. Every extended size-change graph is labelled by a one-element sequence $(F(s_1, ..., s_n) \to G(t_1, ..., t_m))$.*

Concatenation of extended size-change graphs to extended multigraphs works as in Def. 3. However, if G is a multigraph labelled with $(D_1, \ldots, D_n)$ and H is labelled with $(D'_1, \ldots, D'_m)$, then they can only be concatenated if there is an arc from D_n to D'_1 in the (innermost) dependency graph. The concatenation $G \bullet H$ is labelled with $(D_1, \ldots, D_n, D'_1, \ldots, D'_m)$.

In the remainder, when we speak of size-change graphs or multigraphs, we always mean *extended* graphs. To combine dependency pairs and the size-change principle now we only regard multigraphs labelled with a cycle $\mathcal{P}$ of the (innermost) dependency graph (i.e., they are labelled with $(D_1, \ldots, D_n)$ such that $\mathcal{P} = \{D_1, \ldots, D_n\}$). Moreover, one may use different reduction pairs for the multigraphs resulting from different cycles. To benefit from the advantages of the size-change principle (i.e., combining lexicographic and multiset comparison and using different argument filterings and strict inequalities within one cycle), we do not build inequalities but size-change graphs out of the dependency pairs.

The following theorem combines dependency pairs and the size-change principle for full termination (Thm. 11). In contrast to Thm. 11 we now allow arbitrary reduction pairs. However, to handle defined symbols properly, one then has to require that all rules are weakly decreasing (like in the dependency pair approach). Alternatively, as in Thm. 11 one may also use reduction pairs $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$ for a terminating non-duplicating TRS $\mathcal{S}$ over the constructors of $\mathcal{R}$ without requiring that $\mathcal{R}$'s rules are weakly decreasing. For example, in this way one can prove termination of the Ackermann TRS with the embedding ordering (i.e., $\mathcal{S} = Emb_{\mathcal{C}}$). However, in order to use $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$ for some cycles and other reduction pairs $(\succsim, \succ)$ for other cycles, one has to prove termination of $\mathcal{R} \cup \mathcal{S}$ instead of just $\mathcal{R}$.

Example 16. Let $\mathcal{R} = \{\mathsf{g}(\mathsf{f}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{b})), \mathsf{f}(\mathsf{b}) \to \mathsf{f}(\mathsf{a})\}$ and $\mathcal{S} = \{\mathsf{a} \to \mathsf{b}\}$. For the only cycle $\{\mathsf{G}(\mathsf{f}(\mathsf{a})) \to \mathsf{G}(\mathsf{f}(\mathsf{b}))\}$ of $\mathcal{R}$'s dependency graph, size-change termination can be shown by $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$. So if one only regards $\mathcal{R}$ instead of $\mathcal{R} \cup \mathcal{S}$, one could falsely "prove" termination of $\mathcal{R}$. Instead, $\{\mathsf{F}(\mathsf{b}) \to \mathsf{F}(\mathsf{a})\}$ must also be regarded, since it is an $\mathcal{R}$-cycle of the dependency graph of $\mathcal{R} \cup \mathcal{S}$ (in $\mathcal{R} \cup \mathcal{S}$, a is a defined symbol). Moreover, for reduction pairs $(\succsim, \succ) \neq (\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$, one has to demand $l \succsim r$ not only for the rules $l \to r$ of $\mathcal{R}$, but for those of $\mathcal{S}$ as well. Otherwise, the constraints for the cycle $\{\mathsf{F}(\mathsf{b}) \to \mathsf{F}(\mathsf{a})\}$ would falsely be satisfiable.

By Thm. 17, the resulting termination criterion is sound, complete, and more powerful than the size-change principle or dependency pairs on their own.

Theorem 17 (Termination Proofs). *Let $\mathcal{R}$ be a TRS over $\mathcal{F}$ with constructors $\mathcal{C}$ and let $\mathcal{S}$ be a terminating non-duplicating TRS over $\mathcal{C}$. $\mathcal{R}$ (and even $\mathcal{R} \cup \mathcal{S}$) is terminating iff for each $\mathcal{R}$-cycle $\mathcal{P}$ in the dependency graph of $\mathcal{R} \cup \mathcal{S}$ there is a monotonic reduction pair $(\succsim, \succ)$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\sharp, \mathcal{V})$ such that*

(a) all maximal multigraphs w.r.t. $(\succsim, \succ)$ labelled with $\mathcal{P}$ contain an edge $i \xrightarrow{\succsim} i$
(b) $\succsim = \to_{\mathcal{S}}^*$ and $\succ = \to_{\mathcal{S}}^+$ or $l \succsim r$ for all $l \to r \in \mathcal{R} \cup \mathcal{S}$

If $\mathcal{R}$ is size-change terminating w.r.t. $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$ as in Thm. 11 or if a reduction pair satisfies Conditions (a) and (b) of Thm. 14 for termination with dependency pairs, then this reduction pair also satisfies the conditions of this criterion.

Proof. Thm. 17 simulates size-change termination (Thm. 11): If all maximal multigraphs contain $i \xrightarrow{\succsim} i$, this also holds for maximal multigraphs labelled with $\mathcal{P}$. It simulates dependency pairs by choosing $\mathcal{S} = \varnothing$: By Thm. 14 (a), multigraphs labelled with $\mathcal{P}$ contain $\epsilon \xrightarrow{\succsim} \epsilon$. As dependency pairs are *complete* for termination (even with estimated or no dependency graphs), this proves the "only if" part.

For the "if" direction, suppose that $\mathcal{R} \cup \mathcal{S}$ is not terminating. Since $\mathcal{S}$ terminates, by Lemma 10 and the soundness of dependency pairs, there is an infinite chain $s_1 \to t_1, s_2 \to t_2, \ldots$ of $\mathcal{R}$-dependency pairs such that $t_i\sigma \to_{\mathcal{R}\cup\mathcal{S}}^* s_{i+1}\sigma$ for all i and a substitution σ, and $s_1 = s^\sharp$ for a minimal non-terminating term s w.r.t. $\mathcal{R} \cup \mathcal{S}$. Moreover, there is an $\mathcal{R}$-cycle $\mathcal{P}$ consisting of those dependency pairs which occur infinitely often in this chain. Let $i_1 < i_2 < \ldots$ such that $\{s_{i_j} \to t_{i_j}, \ldots, s_{i_{j+1}-1} \to t_{i_{j+1}-1}\} = \mathcal{P}$ for all j, i.e., we partition the sequence into parts where all dependency pairs of $\mathcal{P}$ occur. For all j, let G_j be the multigraph resulting from the concatenation of the size-change graphs corresponding to $s_{i_j} \to t_{i_j}, \ldots, s_{i_{j+1}-1} \to t_{i_{j+1}-1}$. Note that all G_j are labelled with $\mathcal{P}$.

Due to (a), every multigraph H resulting from concatenation of size-change graphs contains an edge of the form $i \xrightarrow{\succ} i$, provided that $H = H \bullet H$ and that H is labelled with $\mathcal{P}$. Hence, every idempotent multigraph $H = H \bullet H$ resulting from concatenating graphs from $G_1, G_2, \ldots$ also contains an edge $i \xrightarrow{\succ} i$. The reason is that since all G_j are labelled with $\mathcal{P}$, then H is also labelled with $\mathcal{P}$.

From this, [13, Thm. 4] or [14, Lemma 7] implies that there is an infinite path with infinitely many "$\succ$"-edges in the infinite graph resulting from $G_1, G_2, \ldots$ by identifying the input nodes of G_j with the output nodes of G_{j+1}. Hence, there is also such a path in the infinite graph resulting from the size-change graphs corresponding to $s_1 \to t_1, s_2 \to t_2, \ldots$ Without loss of generality, we assume that the infinite path already starts in the size-change graph corresponding to $s_1 \to t_1$. For every i, let a_i be the output node in the size-change graph of $s_i \to t_i$ which is on this path. For infinitely many i we have $s_i|_{a_i}\sigma \succ t_i|_{a_{i+1}}\sigma$ and otherwise, we have $s_i|_{a_i}\sigma \succsim t_i|_{a_{i+1}}\sigma$, since $\succsim$ and $\succ$ are closed under substitutions.

If the reduction pair $(\succsim, \succ)$ is $(\to_{\mathcal{S}}^*, \to_{\mathcal{S}}^+)$, then we obtain a contradiction to the minimality of s similar as in the proof of Thm. 11. Otherwise, $t_i|_{a_{i+1}}\sigma \succsim s_{i+1}|_{a_{i+1}}\sigma$ due to (b) since $t_i|_{a_{i+1}}\sigma \to_{\mathcal{R}\cup\mathcal{S}}^* s_{i+1}|_{a_{i+1}}\sigma$. Hence, we have an infinite decreasing sequence w.r.t. $\succ$ which contradicts its well-foundedness. ☐

For innermost termination, we integrate Thm. 8 with dependency pairs. (Integrating a variant of Thm. 11 for innermost termination would only prove innermost termination of $\mathcal{R} \cup \mathcal{S}$ which does not imply innermost termination of $\mathcal{R}$.) In the dependency pair approach for innermost termination, only the *usable* rules for defined symbols in right-hand sides t of dependency pairs $s \to t$ must be weakly decreasing. Here, one can benefit from the size-change principle, which restricts the comparison of terms to certain arguments. Symbols of t which do not occur in the arguments being compared do not have to be regarded as "usable". More precisely, if one uses the extension of a reduction pair which only compares terms with defined symbols from a subset $\mathcal{D}' \subseteq \mathcal{D}$, then one only has to require weak decreasingness of $\mathcal{U}(\mathcal{D}')$. So here the size-change principle has the advantage that one can reduce the set of usable rules.

For example, the Ackermann TRS has the rule $\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$ and therefore, we obtain the dependency pair $\mathsf{ACK}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ACK}(x, \mathsf{ack}(\mathsf{s}(x), y))$. Since ack occurs in the right-hand side of this dependency pair, in the dependency pair approach we would have to require $l \succsim r$ for all ack-rules since they would be regarded as being usable. For this reason, we would need a lexicographic comparison. However, in our new technique, the ACK-dependency pairs are transformed into size-change graphs and size-change termination can easily be shown using the embedding ordering on constructor terms (i.e., $\mathcal{D}' = \varnothing$). In other words, the second argument of $\mathsf{ACK}(x, \mathsf{ack}(\mathsf{s}(x), y))$ is never regarded in this comparison and therefore, the ack-rules are no longer usable. So instead of LPO we only need the embedding ordering to satisfy the resulting constraints. Hence, in the combined technique one can often use much simpler reduction pairs than the reduction pairs needed with dependency pairs.

Here it is important that extensions are non-monotonic. Consider the TRS of Ex. 16 and a reduction pair on constructor terms (i.e., $\mathcal{D}' = \varnothing$) where a is greater than b. Hence, we do not have to regard any usable rules. In the extension $(\succsim, \succ)$ of this reduction pair we have $\mathsf{f}(\mathsf{a}) \not\succ \mathsf{f}(\mathsf{b})$. Thus, the dependency pair $\mathsf{G}(\mathsf{f}(\mathsf{a})) \to \mathsf{G}(\mathsf{f}(\mathsf{b}))$ is not decreasing, i.e., innermost termination is not proved. But if the extension were monotonic, we would falsely prove innermost termination of $\mathcal{R}$.

Theorem 18 (Innermost Termination Proofs). *A TRS $\mathcal{R}$ is innermost terminating if for each cycle $\mathcal{P}$ in the innermost dependency graph there is a reduction pair on $\mathcal{T}(\mathcal{C} \cup \mathcal{D}' \cup \mathcal{F}^\sharp, \mathcal{V})$ for some $\mathcal{D}' \subseteq \mathcal{D}$ which is monotonic if $\mathcal{D}' \neq \varnothing$, such that for its extension $(\succsim, \succ)$ to $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\sharp, \mathcal{V})$ we have*

(a) all maximal multigraphs w.r.t. $(\succsim, \succ)$ labelled with $\mathcal{P}$ contain an edge $i \xrightarrow{\succ} i$
(b) $l \succsim r$ for all $l \to r \in \mathcal{U}(\mathcal{D}')$

If $\mathcal{R}$ is size-change terminating w.r.t. a reduction pair as in Thm. 8 or if a reduction pair satisfies Conditions (c) and (d) of Thm. 14 for innermost termination with dependency pairs, then it also satisfies the conditions of this criterion.

Proof. Thm. 18 can simulate the size-change principle: As in Thm. 17, size-change termination implies (a). Moreover, if $(\succsim, \succ)$ is the extension of a reduction pair on $\mathcal{T}(\mathcal{C}, \mathcal{V})$ as in Thm. 8, then $\mathcal{D}' = \varnothing$ and thus, (b) is also satisfied.

The simulation of dependency pairs and the soundness of the above criterion are shown as for Thm. 17. If $\mathcal{R}$ is not innermost terminating, then there is an infinite innermost chain $s_1 \to t_1, s_2 \to t_2, \dots$ with $t_i\sigma \xrightarrow{\mathrm{i}}{}^*_{\mathcal{R}} s_{i+1}\sigma$ and all $s_i\sigma$ are normal forms. As in Thm. 17's proof, this implies that in the infinite graph resulting from the corresponding size-change graphs there is an infinite path with infinitely many "$\succ$" labels. For every i, let a_i be the output node in the size-change graph corresponding to $s_i \to t_i$ which is on this infinite path. To conclude $t_i|_{a_{i+1}}\sigma \succsim s_{i+1}|_{a_{i+1}}\sigma$, note that $s_i|_{a_i} \succsim t_i|_{a_{i+1}}$ or $s_i|_{a_i} \succ t_i|_{a_{i+1}}$. According to the definition of extending reduction pairs, all subterms of $t_i|_{a_{i+1}}$ with root from $\mathcal{D} \setminus \mathcal{D}'$ also occur in $s_i|_{a_i}$. Hence, when instantiated by σ they are in normal form. Therefore, the only rules applicable to $t_i|_{a_{i+1}}\sigma$ are from $\mathcal{U}(\mathcal{D}')$. Moreover, above the redexes of $t_i|_{a_{i+1}}\sigma$ there are no symbols from $\mathcal{D} \setminus \mathcal{D}'$, since otherwise these redexes would also occur in the normal form $s_i|_{a_i}\sigma$. Now (b) ensures $t_i|_{a_{i+1}}\sigma \succsim s_{i+1}|_{a_{i+1}}\sigma$. The remainder is as in Thm. 17's proof. ☐

The combined technique handles TRSs where both original techniques fail, since some rules require lexicographic or multiset comparison and others require polynomial orderings. In the combined technique, lexicographic or multiset comparison is implicit since the size-change principle is incorporated. Thus, the resulting constraints are often satisfied by simple polynomial orderings. For example, we unite the plus-TRS (Ex. 13) with the TRS for Ackermann's function, where $\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$ is replaced by $\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{plus}(y, \mathsf{ack}(\mathsf{s}(x), y)))$. In the original dependency pair approach, both the ack- and plus-rules are usable for the corresponding dependency pair and thus, no standard ordering amenable to automation fulfills the resulting constraints. But in the combined technique, there are no usable rules and hence, the innermost termination proof works with the simple polynomial ordering on constructors and tuple symbols where $\mathsf{s}(x)$ is mapped to $x + 1$ and $\mathsf{PLUS}(x, y)$ is mapped to $x + y$. In practice, there are many TRSs where the combined technique simplifies the termination proof significantly (e.g., TRSs for arithmetic operations, for sorting algorithms, for term manipulations in λ-calculus, etc., cf. [14]).

In [1, 7], refinements to manipulate dependency pairs by narrowing, rewriting, and instantiation were proposed. These refinements directly carry over to our combined technique. To summarize, the combination of dependency pairs and the size-change principle has two main advantages: First, one can now prove (innermost) termination of TRSs automatically where up to now an automated proof was impossible. Second, for many TRSs where up to now the termination proof required complicated reduction pairs involving a large search space, one can now use much simpler orderings which increases efficiency.

6 Conclusion

We extended the size-change principle to prove (innermost) termination of arbitrary TRSs. Then we compared it with classical simplification orderings from rewriting: It is also restricted to simple termination, it incorporates lexicographic and multiset comparison for root symbols (although not below the root), but it cannot handle defined symbols or term measures and weights. Nevertheless, there are even examples where the size-change principle is advantageous to dependency pairs, since it can simulate argument filtering for root symbols and it can investigate how the size of arguments changes in sequences of function calls. On the other hand, the size-change principle is not modular and it lacks a concept like the dependency graph to analyze which function calls can follow each other. Therefore, we developed a new approach to combine the size-change principle with dependency pairs. The combined approach is more powerful than both previous techniques and has the advantage that it often succeeds with much simpler argument filterings and base orderings than the dependency pair approach. We have implemented both the original dependency pair approach and the combined approach in the system AProVE and found that this combination often increases efficiency dramatically. With this combination and a reduction pair based on the lexicographic path ordering, 103 of the 110 examples in the collection of [2] could be proved innermost terminating fully automatically. Most of these proofs took less than a second; the longest took about 10 seconds. The remaining 7 examples only fail because of the underlying reduction pair (e.g., one would need polynomial orderings or KBO). For details on the experiments see [14].

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