Thm 2.1.16 (Completeness of the Function Domains) *Let D_1, D_2 be domains with cpo's $\sqsubseteq_{D_1}$ and $\sqsubseteq_{D_2}$, respectively. Then $\langle D_1 \to D_2 \rangle$ is the corresponding function domain and $\sqsubseteq_{D_1 \to D_2}$ is a cpo on $\langle D_1 \to D_2 \rangle$.*

Proof. The smallest element of $\langle D_1 \to D_2 \rangle$ is $\perp_{D_1 \to D_2}$. This function is indeed continuous (since every constant function is continuous).

Now let S be a chain in $\langle D_1 \to D_2 \rangle$. Since $\sqsubseteq_{D_1 \to D_2}$ is complete on the set of *all* functions from D_1 to D_2 (Thm. 2.1.13 (c)), there exists a least upper bound $\sqcup S$ which is a function from D_1 to D_2. We have to show that this function is continuous (this means that it is in $\langle D_1 \to D_2 \rangle$).

So we have to show that $(\sqcup S)(\sqcup T) = \sqcup(\sqcup S)(T)$ holds for every chain T from D_1. We obtain

$$
\begin{array}{rcll}
(\sqcup S)(\sqcup T) & = & \sqcup S_{\sqcup T} & \text{by Lemma 2.1.11 (b)} \\
 & = & \sqcup\{f(\sqcup T) | f \in S\} & \\
 & = & \sqcup\{\sqcup f(T) | f \in S\} & \text{since all } f \in S \text{ are continuous} \\
 & = & \sqcup\{\sqcup\{f(x) | x \in T\} | f \in S\} & \\
 & = & \sqcup\{f(x) | x \in T, f \in S\} & \\
 & = & \sqcup\{\sqcup\{f(x) | f \in S\} | x \in T\} & \\
 & = & \sqcup\{\sqcup S_x | x \in T\} & \\
 & = & \sqcup\{(\sqcup S)(x) | x \in T\} & \text{by Lemma 2.1.11 (b)} \\
 & = & \sqcup(\sqcup S)(T) &
\end{array}
$$

The step

$$\sqcup\{\sqcup\{f(x)|x \in T\}|f \in S\} \;=\; \sqcup\{f(x)|x \in T, f \in S\}$$

can be proved as follows:

We first show that $\sqcup\{\sqcup\{f(x)|x \in T\}|f \in S\} \;\sqsubseteq\; \sqcup\{f(x)|x \in T, f \in S\}$, i.e., $\sqcup\{f(x)|x \in T, f \in S\}$ is an upper bound for $\{\sqcup\{f(x)|x \in T\}|f \in S\}$. This is obvious, since $\sqcup\{f(x)|x \in T, f \in S\}$ is greater or equal than all $f(x)$ with $x \in T, f \in S$. Thus, it is an upper bound to $\{f(x)|x \in T\}$ (for arbitrary $f \in S$) and therefore, we obtain $\sqcup\{f(x)|x \in T\} \sqsubseteq \sqcup\{f(x)|x \in T, f \in S\}$.

Now we show $\sqcup\{f(x)|x \in T, f \in S\} \sqsubseteq \sqcup\{\sqcup\{f(x)|x \in T\}|f \in S\}$, i.e., that $\sqcup\{\sqcup\{f(x)|x \in T\}|f \in S\}$ is an upper bound for all $f(x)$ with $x \in T, f \in S$. Clearly, for all such x and f we have $f(x) \sqsubseteq \sqcup\{f(x)|x \in T\} \sqsubseteq \sqcup\{\sqcup\{f(x)|x \in T\}|f \in S\}$.

The step

$$\sqcup\{f(x)|x \in T, f \in S\} \;=\; \sqcup\{\sqcup\{f(x)|f \in S\}|x \in T\}$$

is shown in an analogous way. $\square$