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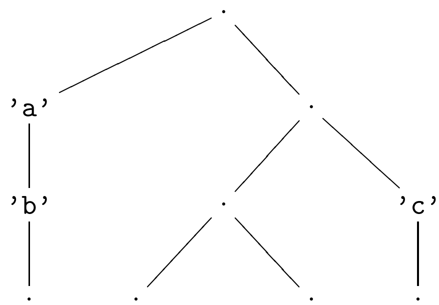
Exercise 1 (4 + 5 + 4 + 6 + 5 = 24 points)

The following data structure represents polymorphic binary trees that contain values only in special `Value` nodes that have a single successor:

```
data Tree a = Leaf | Node (Tree a) (Tree a) | Value a (Tree a)
```

Consider the tree `t` of characters on the right-hand side. The representation of `t` as an object of type `Tree Char` in Haskell would be:




```
(Node (Value 'a' (Value 'b' Leaf)) (Node (Node
Leaf Leaf) (Value 'c' Leaf)))
```

Implement the following functions in Haskell.

(a) The function `foldTree` of type

```
(a -> b -> b) -> (b -> b -> b) -> b -> Tree a -> b
```

works as follows: `foldTree f g h x` replaces all occurrences of the constructor `Value` in the tree `x` by `f`, it replaces all occurrences of the constructor `Node` in `x` by `g`, and it replaces all occurrences of the constructor `Leaf` in `x` by `h`. So for the tree `t` above,

```
foldTree (:) (++) [] t
```

should compute

```
((++) ((:) 'a' ((:) 'b' [])) ((++) ((++) [] []) ((:) 'c' []))),
```

which in the end results in `"abc"` (i.e., in the list `['a','b','c']`). Here, `Value` is replaced by `(:)`, `Node` is replaced by `(++)`, and `Leaf` is replaced by `[]`.

```
foldTree f g h (Value n x) = f n (foldTree f g h x)
foldTree f g h (Node x y)  = g (foldTree f g h x) (foldTree f g h y)
foldTree _ _ h Leaf        = h
```

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(b) Use the `foldTree` function from (a) to implement the `average` function which has the type `Tree Int -> Int` and returns the average of the values that are stored in the tree. This should be accomplished as follows:

- Use `foldTree` with suitable functions as arguments in order to compute the *sum* of the values stored in the trees.
- Use `foldTree` with suitable functions as arguments in order to compute the *number of* `Value`*-objects in the tree.*
- Perform integer division with the pre-defined function `div :: Int -> Int -> Int` on these values to obtain the result.

Here your function is required to work correctly only on those trees that contain the constructor `Value` at least once.

```
average t = div (foldTree (+) (+) 0 t) (foldTree (\x y -> y+1) (+) 0 t)
```

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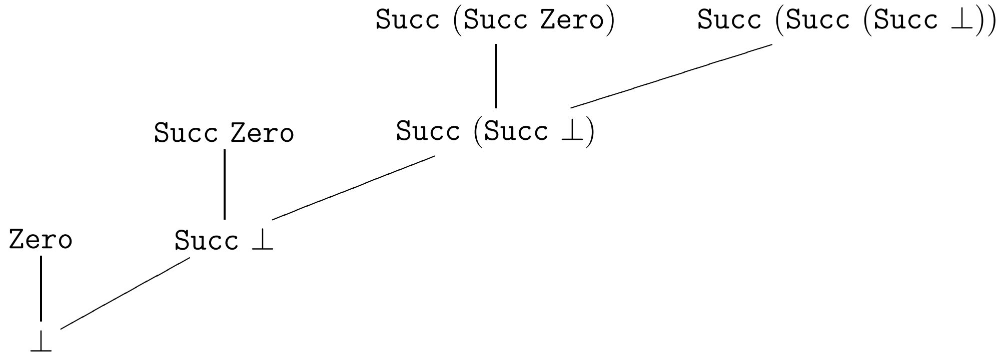
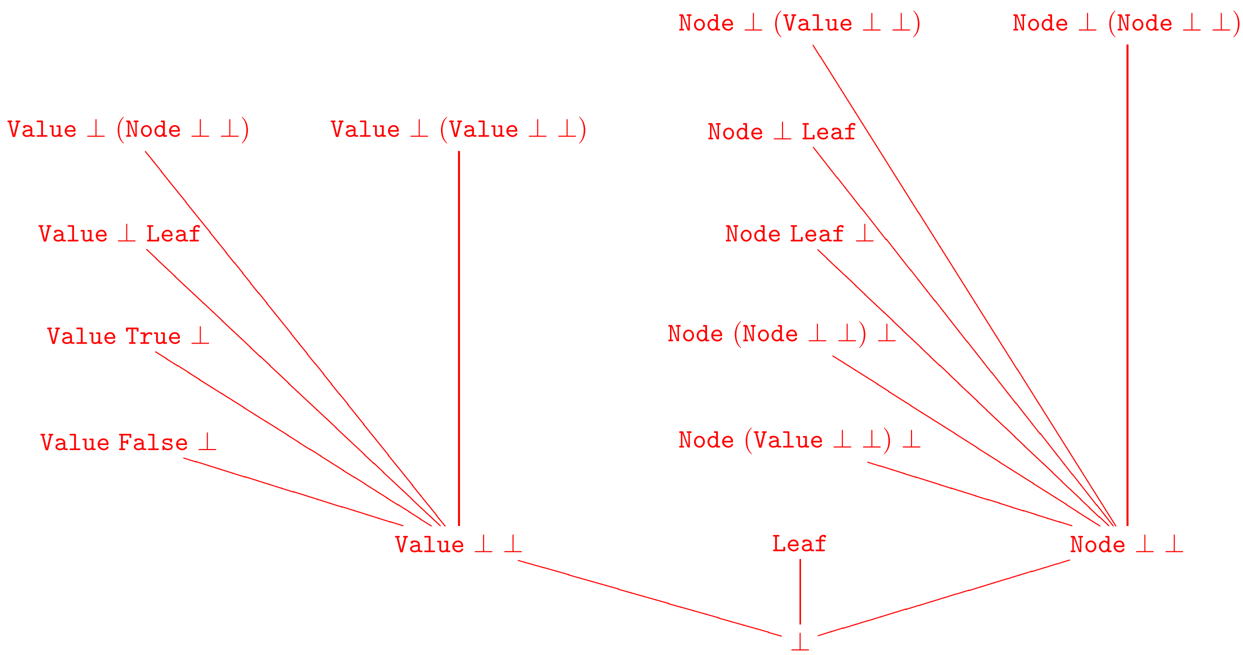
(c) Consider the following data type declaration for natural numbers:

```
data Nats = Zero | Succ Nats
```

A graphical representation of the first four levels of the domain for `Nats` could look like this:



Sketch a graphical representation of the first three levels of the domain for the data type `Tree Bool`.



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(d) We call a list `ys` of integers an `n`-*times even product* of a list `xs` if `ys` has length `n` and if all elements of `ys` are even numbers that occur in `xs`. The goal of this exercise is to write a function `evenProducts :: [Int] -> Int -> [[Int]]` that takes a list of integers `xs` and a natural number `n` and returns a list that contains all `n`-times even products of `xs`. For example, `evenProducts [4,5,6] 2 = [[4,4], [4,6], [6,4], [6,6]]`.

The following declarations are already given:

```
evenProducts :: [Int] -> Int -> [[Int]]
evenProducts xs 0     = []
evenProducts xs 1     = map (\z -> [z]) (filter even xs)
```

Please give the declaration of `evenProducts` for the missing case of numbers that are at least 2. Perform your implementation only with the help of a **list comprehension**, i.e., you should use exactly one declaration of the following form:

```
evenProducts xs (n+2) = [ ... | ... ]
```

```
evenProducts xs (n+2) = [ y:ys | y <- xs, even y, ys <- evenProducts xs (n+1)]
```

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(e) We define the *level* n of a tree as the list of those values that are at distance n from the root of the tree. Here, the root node has distance 0 from the root, and a non-root node has distance $n + 1$ from the root if its parent node has distance n from the root.

Write a Haskell function `level :: Tree a -> Int -> [a]` which, given a tree `t` and a natural number `n`, computes the list of all values in `t` that occur there at level `n` (with repetition, i.e., a value should appear in the result list as many times as it appears on level `n`).

As an example, consider again the tree `t` from the beginning of the exercise. Here we have `level t 2 = ['b','c']` and `level t 7 = []`.

```
level :: Tree a -> Int -> [a]
level t 0 = case t of Value x u -> [x]
                      _ -> []
level t (n+1) = case t of Leaf -> []
                          Value x u -> level u n
                          Node u v -> (level u n) ++ (level v n)
```

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Exercise 2 (4 + 5 = 9 points)

Consider the following Haskell declarations for the `fib` function, which for a natural number x computes the value $\mathit{fibonacci}(x)$:

```
fib :: Int -> Int
fib 0     = 0
fib 1     = 1
fib (x+2) = fib (x+1) + fib x
```

(a) Please give the Haskell declarations for the higher-order function `f_fib` corresponding to `fib`, i.e., the higher-order function `f_fib` such that the least fixpoint of `f_fib` is `fib`. In addition to the function declaration(s), please also give the type declaration of `f_fib`. Since you may use full Haskell for `f_fib`, you do not need to translate `fib` into simple Haskell.

```
f_fib :: (Int -> Int) -> (Int -> Int)
f_fib fib 0 = 0
f_fib fib 1 = 1
f_fib fib (x+2) = fib (x+1) + fib x
```

(b) We add the Haskell declaration `bot = bot`. For each $n \in \mathbb{N}$ please determine which function is computed by $\texttt{f\_fib}^n$ `bot`. Here "$\texttt{f\_fib}^n$ `bot`" represents the n-fold application of `f_fib` to `bot`, i.e., it is short for $\underbrace{\texttt{f\_fib}\ (\texttt{f\_fib}\ \ldots\ (\texttt{f\_fib}}_{n \text{ times}}\ \texttt{bot})\ldots)$.

Let $f_n : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$ be the function that is computed by $\texttt{f\_fib}^n$ `bot`.
Give f_n in **closed form**, i.e., using a non-recursive definition. In this definition, you may use the function $\mathit{fibonacci} : \mathbb{N} \to \mathbb{N}$ where $\mathit{fibonacci}(x)$ computes the x-th Fibonacci number. Here it suffices to give the result of your calculations. You do not need to present any intermediate steps.

$$(\texttt{f\_fib}^n(\perp))(x) = \begin{cases} \mathit{fibonacci}(x), & \text{if } n > 0 \text{ and } 0 \le x \le n \\ \perp, & \text{otherwise} \end{cases}$$

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Exercise 3 (3 + 3 = 6 points)

Let D_1, D_2, D_3 be domains with corresponding complete partial orders $\sqsubseteq_{D_1}, \sqsubseteq_{D_2}, \sqsubseteq_{D_3}$. As we know from the lecture, then also $\sqsubseteq_{(D_2 \times D_3)_\perp}$ is a complete partial order on $(D_2 \times D_3)_\perp$.

Now let $f : D_1 \to D_2$ and $g : D_1 \to D_3$ be functions.
We then define the function $h : D_1 \to (D_2 \times D_3)_\perp$ via $h(x) = (f(x), g(x))$.

(a) Prove or disprove: If f and g are *strict* functions, then also h is a strict function.

The statement does not hold. Consider the following counterexample: $D_1 = D_2 = D_3 = \mathbb{B}_\perp$ and $f = g = \perp_{\mathbb{B}_\perp \to \mathbb{B}_\perp}$. Obviously f and g are strict functions, i.e., $f(\perp_{\mathbb{B}_\perp}) = g(\perp_{\mathbb{B}_\perp}) = \perp_{\mathbb{B}_\perp}$. However, we have $h(\perp_{\mathbb{B}_\perp}) = (\perp_{\mathbb{B}_\perp}, \perp_{\mathbb{B}_\perp}) \neq \perp_{(\mathbb{B}_\perp \times \mathbb{B}_\perp)_\perp}$.

(b) Prove or disprove: If f and g are *monotonic* functions, then also h is a monotonic function.

Let $x \sqsubseteq_{D_1} y$. Then we have:

$$\begin{array}{cll} & h(x) & \\ = & (f(x), g(x)) & \quad f \text{ and } g \text{ are monotonic, def. of } \sqsubseteq_{(D_2 \times D_3)_\perp} \\ \sqsubseteq_{(D_2 \times D_3)_\perp} & (f(y), g(y)) & \\ = & h(y) & \end{array}$$

Hence, also h is monotonic. $\square$

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Exercise 4 (4 + 5 = 9 points)

Consider the following data structure for polymorphic lists:

```
data List a = Nil | Cons a (List a)
```

(a) Please translate the following Haskell expression into an equivalent lambda term (e.g., using $\mathcal{L}am$). Recall that pre-defined functions like `odd` or `(+)` are translated into constants of the lambda calculus.

It suffices to give the result of the transformation.

```
let f = \x -> if (odd x) then (\y -> x) else f ((+) x 3)
    in f
```

$$\texttt{fix}\ (\lambda f\ x.\ \texttt{if}\ \ (\texttt{odd}\ x)\ \ (\lambda y.x)\ \ (f\ ((\texttt{+})\ x\ 3))\ )$$

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(b) Let δ be the set of rules for evaluating the lambda terms resulting from Haskell, i.e., δ contains at least the following rules:

$$\begin{aligned} \texttt{fix} &\to \lambda f.\, f\ (\texttt{fix}\ f) \\ \texttt{times}\ 3\ 2 &\to 6 \end{aligned}$$

Now let the lambda term t be defined as follows:

$$t = (\lambda x.\, (\texttt{fix}\ \ \lambda g.\, x))\ \ (\lambda z.\, (\texttt{times}\ 3\ 2))$$

Please reduce the lambda term t by WHNO-reduction with the $\to_{\beta\delta}$-relation. You have to give **all** intermediate steps until you reach **weak head normal form** (and no further steps).

$$\begin{aligned} & (\lambda x.\, (\texttt{fix}\ \ \lambda g.\, x))\ \ (\lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\beta\ & \texttt{fix}\ \ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\delta\ & (\lambda f.\, f\ \ (\texttt{fix}\ f))\ \ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\beta\ & (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2))\ (\texttt{fix}\ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2))) \\ \to_\beta\ & \lambda z.\, (\texttt{times}\ 3\ 2) \end{aligned}$$

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Exercise 5 (10 points)

Use the type inference algorithm $\mathcal{W}$ to determine the most general type of the following lambda term under the initial type assumption A_0. Show the results of all sub-computations and unifications, too. If the term is not well typed, show how and why the $\mathcal{W}$-algorithm detects this.

$$(\,(\texttt{Cons}\ \ \lambda x.\, x)\ y\,)$$

The initial type assumption A_0 contains at least the following:

$$\begin{array}{lcl} A_0(\texttt{Cons}) & = & \forall a.\ (a \to (\texttt{List}\ a \to \texttt{List}\ a)) \\ A_0(x) & = & \forall a.\ a \\ A_0(y) & = & \forall a.\ a \end{array}$$

$\mathcal{W}(A_0,\ (\,(\texttt{Cons}\ \ \lambda x.\, x)\ y\,)\ )$

$\quad\mathcal{W}(A_0,\ (\texttt{Cons}\ \ \lambda x.\, x)\ )$

$\quad\quad\mathcal{W}(A_0,\ \texttt{Cons})$

$\quad\quad= (id,\ (b_1 \to\ (\texttt{List}\ b_1 \to\ \texttt{List}\ b_1))\ )$

$\quad\quad\mathcal{W}(A_0,\ \lambda x.\, x\ )$

$\quad\quad\quad\mathcal{W}(A_0 + \{x :: b_2\},\ x\ )$

$\quad\quad\quad= (id,\ b_2)$

$\quad\quad= (id, (b_2 \to b_2))$

$\quad\quad mgu(\ (b_1 \to\ (\texttt{List}\ b_1 \to \texttt{List}\ b_1)),\ ((b_2 \to b_2) \to b_3)\ )$

$\quad\quad\quad\quad= [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2))\,]$

$\quad= (\ [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2))\ ],\ (\texttt{List}\ (b_2 \to b_2) \to\ \texttt{List}\ (b_2 \to b_2))\ )$

$\quad\mathcal{W}(A_0,\ y)$

$\quad= (id,\ b_4)$

$\quad mgu(\ (\texttt{List}\ (b_2 \to b_2) \to\ \texttt{List}\ (b_2 \to b_2)),\ (b_4 \to b_5)\ ) = [\ b_4/\texttt{List}\ (b_2 \to b_2),\ b_5/\texttt{List}\ (b_2 \to b_2)\ ]$

$= (\ [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2)),\ b_4/\texttt{List}\ (b_2 \to b_2),\ b_5/\texttt{List}\ (b_2 \to b_2)\ ],$
$\quad\quad\texttt{List}\ (b_2 \to b_2)\ )$

Resulting type: $\texttt{List}\ (b_2 \to b_2)$