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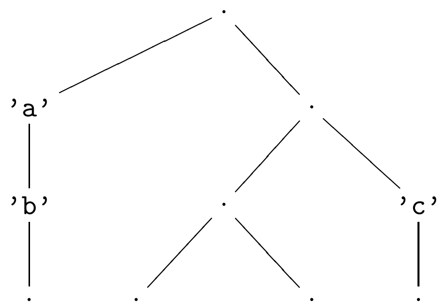
Exercise 1 ($4 + 5 + 4 + 5 = 18$ points)

The following data structure represents polymorphic binary trees that contain values only in special `Value` nodes that have a single successor:

```
data Tree a = Leaf | Node (Tree a) (Tree a) | Value a (Tree a)
```

Consider the tree `t` of characters on the right-hand side. The representation of `t` as an object of type `Tree Char` in Haskell would be:

```
(Node (Value 'a' (Value 'b' Leaf)) (Node (Node
Leaf Leaf) (Value 'c' Leaf)))
```



Implement the following functions in Haskell.

(a) The function `foldTree` of type

```
(a -> b -> b) -> (b -> b -> b) -> b -> Tree a -> b
```

works as follows: `foldTree f g h x` replaces all occurrences of the constructor `Value` in the tree `x` by `f`, it replaces all occurrences of the constructor `Node` in `x` by `g`, and it replaces all occurrences of the constructor `Leaf` in `x` by `h`. So for the tree `t` above,

```
foldTree (:) (++) [] t
```

should compute

```
((++) ((:) 'a' ((:) 'b' [])) ((++) ((++) [] []) ((:) 'c' []))),
```

which in the end results in `"abc"` (i.e., in the list `['a','b','c']`). Here, `Value` is replaced by `(:)`, `Node` is replaced by `(++)`, and `Leaf` is replaced by `[]`.

```
foldTree f g h (Value n x) = f n (foldTree f g h x)
foldTree f g h (Node x y)  = g (foldTree f g h x) (foldTree f g h y)
foldTree _ _ h Leaf        = h
```

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(b) Use the `foldTree` function from (a) to implement the `average` function which has the type `Tree Int -> Int` and returns the average of the values that are stored in the tree. This should be accomplished as follows:

- Use `foldTree` with suitable functions as arguments in order to compute the *sum* of the values stored in the trees.
- Use `foldTree` with suitable functions as arguments in order to compute the *number of* `Value`*-objects in the tree.*
- Perform integer division with the pre-defined function `div :: Int -> Int -> Int` on these values to obtain the result.

Here your function is required to work correctly only on those trees that contain the constructor `Value` at least once.

```
average t = div (foldTree (+) (+) 0 t) (foldTree (\x y -> y+1) (+) 0 t)
```

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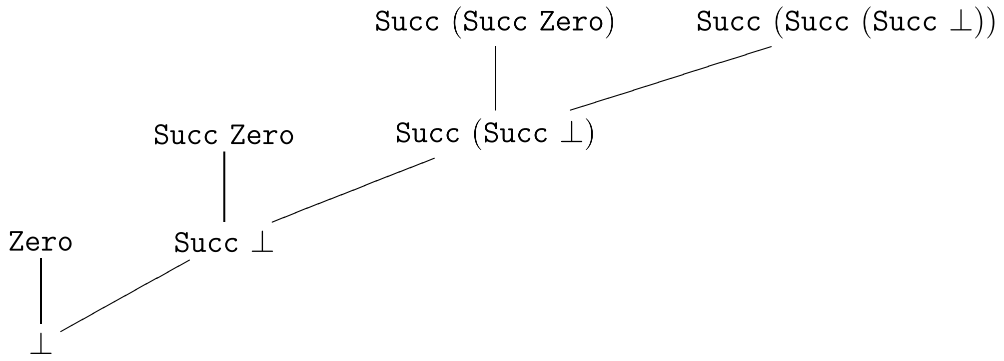
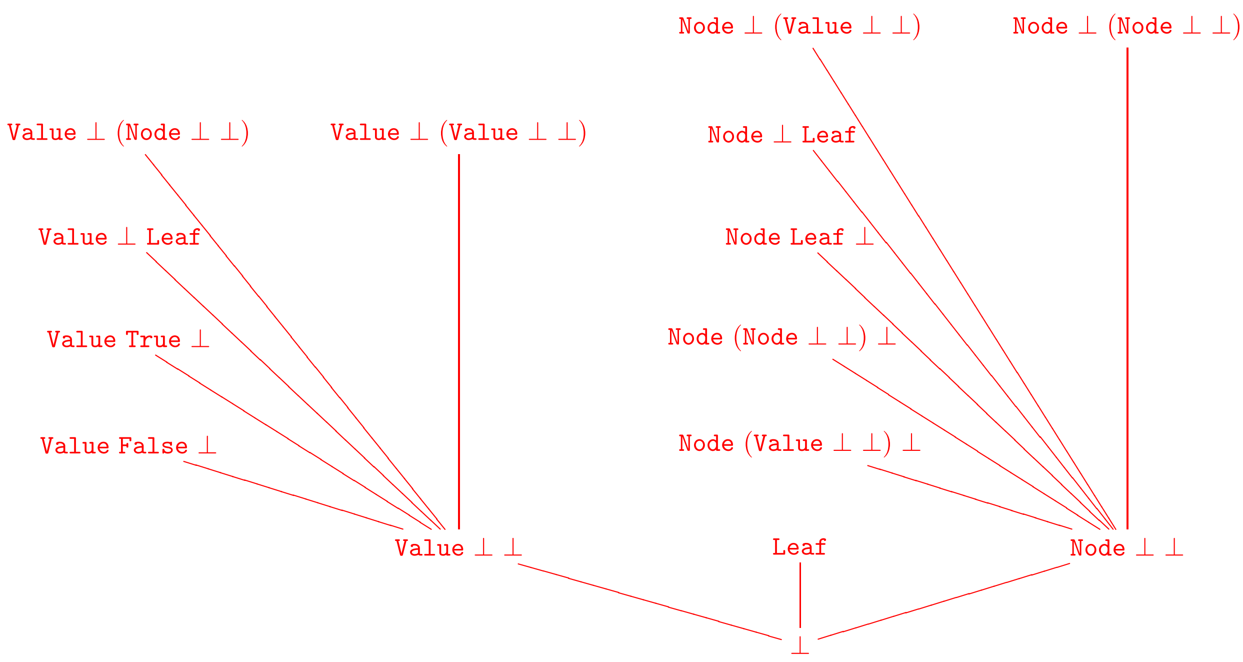
(c) Consider the following data type declaration for natural numbers:

```
data Nats = Zero | Succ Nats
```

A graphical representation of the first four levels of the domain for `Nats` could look like this:



Sketch a graphical representation of the first three levels of the domain for the data type `Tree Bool`.



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(d) Write a Haskell function `printStars` that first reads a string from the user, then prints this string on the console, converts the string to a number `n` (using the pre-defined function `read`) and in the end also prints `n` times the character `'*'` on the console. Also give the type declaration for your function.

You may use the `do`-notation, but you are not obliged to use it. You do not have to check whether the input string is really a number. Some of the following pre-defined functions can be helpful:

- `getLine :: IO String` reads a string from the user
- `read :: String -> Int` converts a string to a number
- `putStr :: String -> IO ()` writes a string to the console

An example run should look as given below. Here the string "7" was read from the user.

```
Main> printStars
7
7*******
```

```
-- without do-notation
printStars :: IO ()
printStars = getLine >>= \s -> putStr s >> putStr (take (read s) (repeat '*'))

-- alternative: with do-notation
printStars2 :: IO ()
printStars2 = do s <- getLine
                 putStr s
                 putStr (take (read s) (repeat '*'))
```

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Exercise 2 (4 + 5 = 9 points)

Consider the following Haskell declarations for the `fib` function, which for a natural number x computes the value $\mathit{fibonacci}(x)$:

```
fib :: Int -> Int
fib 0     = 0
fib 1     = 1
fib (x+2) = fib (x+1) + fib x
```

(a) Please give the Haskell declarations for the higher-order function `f_fib` corresponding to `fib`, i.e., the higher-order function `f_fib` such that the least fixpoint of `f_fib` is `fib`. In addition to the function declaration(s), please also give the type declaration of `f_fib`. Since you may use full Haskell for `f_fib`, you do not need to translate `fib` into simple Haskell.

```
f_fib :: (Int -> Int) -> (Int -> Int)
f_fib fib 0 = 0
f_fib fib 1 = 1
f_fib fib (x+2) = fib (x+1) + fib x
```

(b) We add the Haskell declaration `bot = bot`. For each $n \in \mathbb{N}$ please determine which function is computed by $\texttt{f\_fib}^n$ `bot`. Here "$\texttt{f\_fib}^n$ `bot`" represents the n-fold application of `f_fib` to `bot`, i.e., it is short for $\underbrace{\texttt{f\_fib (f\_fib} \ldots \texttt{(f\_fib}}_{n \text{ times}} \texttt{ bot)} \ldots)$.

Let $f_n : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$ be the function that is computed by $\texttt{f\_fib}^n$ `bot`.
Give f_n in **closed form**, i.e., using a non-recursive definition. In this definition, you may use the function $\mathit{fibonacci} : \mathbb{N} \to \mathbb{N}$ where $\mathit{fibonacci}(x)$ computes the x-th Fibonacci number. Here it suffices to give the result of your calculations. You do not need to present any intermediate steps.

$$(\texttt{f\_fib}^n(\perp))(x) = \begin{cases} \mathit{fibonacci}(x), & \text{if } n > 0 \text{ and } 0 \leq x \leq n \\ \perp, & \text{otherwise} \end{cases}$$

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Exercise 3 (3 + 3 = 6 points)

Let D_1, D_2, D_3 be domains with corresponding complete partial orders $\sqsubseteq_{D_1}, \sqsubseteq_{D_2}, \sqsubseteq_{D_3}$. As we know from the lecture, then also $\sqsubseteq_{(D_2 \times D_3)_\bot}$ is a complete partial order on $(D_2 \times D_3)_\bot$.

Now let $f : D_1 \to D_2$ and $g : D_1 \to D_3$ be functions.
We then define the function $h : D_1 \to (D_2 \times D_3)_\bot$ via $h(x) = (f(x), g(x))$.

(a) Prove or disprove: If f and g are *strict* functions, then also h is a strict function.

The statement does not hold. Consider the following counterexample: $D_1 = D_2 = D_3 = \mathbb{B}_\bot$ and $f = g = \bot_{\mathbb{B}_\bot \to \mathbb{B}_\bot}$. Obviously f and g are strict functions, i.e., $f(\bot_{\mathbb{B}_\bot}) = g(\bot_{\mathbb{B}_\bot}) = \bot_{\mathbb{B}_\bot}$. However, we have $h(\bot_{\mathbb{B}_\bot}) = (\bot_{\mathbb{B}_\bot}, \bot_{\mathbb{B}_\bot}) \neq \bot_{(\mathbb{B}_\bot \times \mathbb{B}_\bot)_\bot}$.

(b) Prove or disprove: If f and g are *monotonic* functions, then also h is a monotonic function.

Let $x \sqsubseteq_{D_1} y$. Then we have:

$$\begin{array}{cll} & h(x) & \\ = & (f(x), g(x)) & f \text{ and } g \text{ are monotonic, def. of } \sqsubseteq_{(D_2 \times D_3)_\bot} \\ \sqsubseteq_{(D_2 \times D_3)_\bot} & (f(y), g(y)) & \\ = & h(y) & \end{array}$$

Hence, also h is monotonic. $\square$

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Exercise 4 (6 points)

We define the following data structures for natural numbers and polymorphic lists:

```
data Nats = Zero | Succ Nats
data List a = Nil | Cons a (List a)
```

Consider the following expression in complex Haskell:

```
let get n        Nil         = Zero
    get Zero     (Cons x xs) = x
    get (Succ n) (Cons x xs) = get n xs
    in get
```

Please give an equivalent expression `let get = ... in get` in **simple** Haskell.

Your solution should use the functions defined in the transformation from the lecture such as $\texttt{sel}_{n,i}$, $\texttt{isa}_{\underline{\texttt{constr}}}$, and $\texttt{argof}_{\underline{\texttt{constr}}}$. However, you do not have to use the transformation rules from the lecture.

```
let get = \n -> \xs -> if (isa_Nil xs)
                       then Zero
                       else if (isa_Zero n)
                            then (sel_2,1 (argof_Cons xs))
                            else get (sel_1,1 (argof_Succ n)) (sel_2,2 (argof_Cons xs))
    in get
```

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Exercise 5 (4 + 5 = 9 points)

Consider the following data structure for polymorphic lists:

```
data List a = Nil | Cons a (List a)
```

(a) Please translate the following Haskell expression into an equivalent lambda term (e.g., using $\mathcal{L}am$). Recall that pre-defined functions like `odd` or `(+)` are translated into constants of the lambda calculus.

It suffices to give the result of the transformation.

```
let f = \x -> if (odd x) then (\y -> x) else f ((+) x 3)
    in f
```

$$\texttt{fix}\ (\lambda f\ x.\ \texttt{if}\ \ (\texttt{odd}\ x)\ \ (\lambda y.x)\ \ (f\ ((\texttt{+})\ x\ \texttt{3}))\ )$$

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(b) Let δ be the set of rules for evaluating the lambda terms resulting from Haskell, i.e., δ contains at least the following rules:

$$\begin{array}{rcl} \texttt{fix} & \to & \lambda f.\, f\ (\texttt{fix}\ f) \\ \texttt{times}\ 3\ 2 & \to & 6 \end{array}$$

Now let the lambda term t be defined as follows:

$$t = (\lambda x.\, (\texttt{fix}\ \ \lambda g.\, x))\ \ (\lambda z.\, (\texttt{times}\ 3\ 2))$$

Please reduce the lambda term t by WHNO-reduction with the $\to_{\beta\delta}$-relation. You have to give **all** intermediate steps until you reach **weak head normal form** (and no further steps).

$$\begin{array}{rl} & (\lambda x.\, (\texttt{fix}\ \ \lambda g.\, x))\ \ (\lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\beta & \texttt{fix}\ \ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\delta & (\lambda f.\, f\ \ (\texttt{fix}\ f))\ \ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2)) \\ \to_\beta & (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2))\ (\texttt{fix}\ (\lambda g.\, \lambda z.\, (\texttt{times}\ 3\ 2))) \\ \to_\beta & \lambda z.\, (\texttt{times}\ 3\ 2) \end{array}$$

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Exercise 6 (10 points)

Use the type inference algorithm $\mathcal{W}$ to determine the most general type of the following lambda term under the initial type assumption A_0. Show the results of all sub-computations and unifications, too. If the term is not well typed, show how and why the $\mathcal{W}$-algorithm detects this.

$$(\,(\texttt{Cons}\ \ \lambda x.\,x)\ y\,)$$

The initial type assumption A_0 contains at least the following:

$$\begin{array}{lcl} A_0(\texttt{Cons}) & = & \forall a.\ (a \to (\texttt{List}\ a \to \texttt{List}\ a)) \\ A_0(x) & = & \forall a.\ a \\ A_0(y) & = & \forall a.\ a \end{array}$$

$\mathcal{W}(A_0,\ (\,(\texttt{Cons}\ \ \lambda x.\,x)\ y\,)\ )$

$\quad\mathcal{W}(A_0,\ (\texttt{Cons}\ \ \lambda x.\,x)\ )$

$\quad\quad\mathcal{W}(A_0,\ \texttt{Cons})$

$\quad\quad= (id,\ (b_1 \to\ (\texttt{List}\ b_1 \to\ \texttt{List}\ b_1))\ )$

$\quad\quad\mathcal{W}(A_0,\ \lambda x.\,x\ )$

$\quad\quad\quad\mathcal{W}(A_0 + \{x :: b_2\},\ x\ )$

$\quad\quad\quad= (id,\ b_2)$

$\quad\quad= (id, (b_2 \to b_2))$

$\quad\quad mgu(\ (b_1 \to\ (\texttt{List}\ b_1 \to \texttt{List}\ b_1)),\ ((b_2 \to b_2) \to b_3)\ )$

$\quad\quad\quad\quad= [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2))\ ]$

$\quad= (\ [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2))\ ],\ (\texttt{List}\ (b_2 \to b_2) \to\ \texttt{List}\ (b_2 \to b_2))\ )$

$\quad\mathcal{W}(A_0,\ y)$

$\quad= (id,\ b_4)$

$\quad mgu(\ (\texttt{List}\ (b_2 \to b_2) \to\ \texttt{List}\ (b_2 \to b_2)),\ (b_4 \to b_5)\ ) = [\ b_4/\texttt{List}\ (b_2 \to b_2),\ b_5/\texttt{List}\ (b_2 \to b_2)\ ]$

$= (\ [\ b_1/(b_2 \to b_2),\ b_3/(\texttt{List}\ (b_2 \to b_2) \to \texttt{List}\ (b_2 \to b_2)),\ b_4/\texttt{List}\ (b_2 \to b_2),\ b_5/\texttt{List}\ (b_2 \to b_2)\ ],$

$\quad\quad\texttt{List}\ (b_2 \to b_2)\ )$

Resulting type: $\texttt{List}\ (b_2 \to b_2)$