Exam in Functional Programming SS 2012 (V3B)

First Name: __________________________________________

Last Name: __________________________________________

Matriculation Number: ______________________________________

Course of Studies (please mark exactly one):

- Informatik Bachelor
- Informatik Master
- Mathematik Master
- Software Systems Engineering Master
- Other: __________________________________________

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Notes:

- **On all sheets** (including additional sheets) you must write **your first name, your last name and your matriculation number**.

- Give your answers in readable and understandable form.

- Use **permanent** pens. Do not use red or green pens and do not use pencils.

- Please write your answers on the exam sheets (also use the reverse sides).

- For each exercise part, give **at most one solution**. Cancel out everything else. Otherwise all solutions of the particular exercise part will give you **0 points**.

- If we observe any **attempt of deception**, the whole exam will be evaluated to **0 points**.

- At the end of the exam, hand in all **sheets together with the sheets containing the exam questions**.
Exercise 1 (Quiz): \((4 + 5 + 5 + 3 + 3 = 20 \text{ points})\)

Give a short proof sketch or a counterexample for each of the following statements:

a) Monotonic unary functions are always strict.

b) Strict unary functions on flat domains are always monotonic.

c) Let \(\mathbb{B}\) be the Boolean values true, false.

Is \(f: (\mathbb{B} \rightarrow \mathbb{B}^\bot) \rightarrow \mathbb{Z}\) with \(f(g) = \begin{cases} 1 & \text{if } g(x) \neq \text{true} \text{ for all } x \in \mathbb{B} \\ 0 & \text{otherwise} \end{cases}\) monotonic?

d) Is \(\rightarrow_\alpha\) terminating?

e) Is \(\rightarrow_\alpha\) confluent?
Exercise 2 (Programming in Haskell):

We define a polymorphic data structure `ZombieHalls` to represent a zombie-infested school whose classrooms contain different types of food:

```haskell
data ZombieHalls food =
    HallwayFork (ZombieHalls food) (ZombieHalls food)
  | HallwayClassroom (Int, food) (ZombieHalls food)
  | HallwayEnd
```

Here, we use three data constructors: One representing the case that the hallway forks and we can go in two directions, one for the case that we have a classroom on one side and can continue in the hallway and finally one case for the end of a hallway. The data structure `ZombieFood` is used to represent food for zombies. As example, consider the following definition of `exampleSchool` of type `ZombieLabyrinth ZombieFood`, corresponding to the illustration on the right:

```haskell
data ZombieFood = Brains | Nuts deriving Show

exampleSchool :: ZombieHalls ZombieFood
exampleSchool =
    HallwayClassroom (3, Nuts)
    (HallwayFork
        (HallwayClassroom (4, Brains)
            (HallwayFork HallwayEnd HallwayEnd))
        (HallwayClassroom (0, Brains) HallwayEnd))
```

a) Implement a function `buildSchool :: Int -> ZombieHalls ZombieFood` such that for any integer number \( n \geq 0 \), it returns a structure of hallways containing \( 2^{n+1} \) classrooms in total. Of these, one half should each contain one brain and the other should each contain one nut.
b) Implement a fold function foldZombieHalls, including its type declaration, for the data structure ZombieHalls. As usual, the fold function replaces the data constructors in a ZombieHalls expression by functions specified by the user. The first argument of foldZombieHalls should be the function for the case of a HallwayFork, the second argument should replace the HallwayClassroom constructor and the third argument should replace the HallwayEnd data constructor. As an example, consider the following function definition, which uses foldZombieHalls to determine the number of dead ends in a ZombieHalls structure, where a classroom does not count as dead end. Hence, the call numberOfDeadEnds exampleSchool returns 3.

```haskell
numberOfDeadEnds :: ZombieHalls food -> Int
numberOfDeadEnds school = foldZombieHalls (+) (_ r -> r) 1 school
```

c) Implement the function bcCounter :: ZombieHalls ZombieFood -> (Int, Int), which counts the number of brains and classrooms in a given school and returns the two numbers as a tuple of integers. The first part of the tuple should be the number of brains in the school and the second should be the number of classrooms. For the definition of bcCounter, use only one defining equation where the right-hand side is just one call to the function foldZombieHalls. However, you may use and define non-recursive auxiliary functions.

For example, a call bcCounter exampleSchool should return the tuple (4, 3).
d) The infinite sequence of Fibonacci numbers \( \text{fib}_i \) is defined as \( \text{fib}_0 = 0 \), \( \text{fib}_1 = 1 \) and \( \text{fib}_i = \text{fib}_{i-1} + \text{fib}_{i-2} \) for all \( i > 1 \). The first elements of the sequence are 0, 1, 1, 2, 3, 5, 8, 13, 21, ...  
Implement a cyclic data structure \( \text{fibs} :: [\text{Int}] \) that represents the infinite list of Fibonacci numbers. Do not use self-defined auxiliary functions and ensure that \( \text{take n fibs} \) has linear complexity.

Hints:

- You should use the function \( \text{zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]} \), which applies the function given as its first argument to combine the elements of two lists. For example \( \text{zipWith (++) ["a","b"] ["c", "d", "e"]} \) results in the list \(["ac","bd"]\). Note that the length of the resulting list is the smallest length of both input lists.
- You may use the pre-defined function \( \text{tail defined as} \, \text{tail (x:xs)} = \text{xs} \).

e) Write a function \( \text{splits :: [a] -> [(a],[a])] \) that computes all splits of a finite input list, i.e., a call \( \text{splits xs} \) should return all pairs \((ys,zs)\) such that \( ys ++ zs \) is again \( xs \). For example, we have \( \text{splits "abc"} = [("","abc")],("a","bc")],("ab","c")],("abc","")\].

The right-hand side of your function should be just a list comprehension.

Hints:

- Use \( \text{length :: [a] -> Int} \), which returns the length of a given list.
- Use \( \text{take :: Int -> [a] -> [a]} \), where \( \text{take n xs} \) yields the longest prefix of \( xs \) with length \( \leq n \).
- Use \( \text{drop :: Int -> [a] -> [a]} \), where \( \text{drop n xs} \) returns the list obtained from \( xs \) by removing the first \( n \) elements.
Exercise 3 (Semantics):

a)  i) Let $\preceq_{D_1}$ and $\preceq_{D_2}$ be complete partial orders on $D_1$ resp. $D_2$ and $f : D_1 \rightarrow D_2$ a function. Prove that $f$ is continuous if and only if $f$ is monotonic and for all chains $S$ in $D_1$, $f(\sqcup S) \subseteq D_2 \sqcup f(S)$ holds.
ii) Let $D = \mathbb{N} \to \{1\}_\perp$, i.e., $D$ is the set of all functions mapping the natural numbers to $\perp$ or $1$. Let \(\sqsubseteq\) be defined as usual on functions.

1) Prove that every chain $S \sqsubseteq D$ has a least upper bound w.r.t. the relation $\sqsubseteq$.

2) Prove that $\sqsubseteq$ is a cpo on $D$.

3) Give an example for an infinite chain in $(D, \sqsubseteq)$.

4) Give a monotonic, non-continuous function $f : D \to D$. You do not need to prove that $f$ has these properties.
b) i) Consider the following Haskell function \( \text{exp} \):

\[
\begin{align*}
\text{exp} :: (\text{Int}, \text{Int}) & \to \text{Int} \\
\text{exp} (x, 0) & = 1 \\
\text{exp} (x, y) & = x \times \text{exp} (x, y - 1)
\end{align*}
\]

Please give the Haskell declaration for the higher-order function \( \text{f}\_\text{exp} \) corresponding to \( \text{exp} \), i.e., the higher-order function \( \text{f}\_\text{exp} \) such that the least fixpoint of \( \text{f}\_\text{exp} \) is \( \text{exp} \). In addition to the function declaration, please also give the type declaration of \( \text{f}\_\text{exp} \). You may use full Haskell for \( \text{f}\_\text{exp} \).

ii) Let \( \phi_\text{f}\_\text{exp} \) be the semantics of the function \( \text{f}\_\text{exp} \). Give the semantics of \( \phi_\text{f}\_\text{exp}^n (\bot) \) for \( n \in \mathbb{N} \), i.e., the semantics of the \( n \)-fold application of \( \phi_\text{f}\_\text{exp} \) to \( \bot \).

iii) Give the least fixpoint of \( \phi_\text{f}\_\text{exp} \).
c) Consider the following data type declaration for natural numbers:

\[
\text{data Nats } = \text{Z } | \text{ S Nats }
\]

A graphical representation of the first four levels of the domain for Nats could look like this:

\[
\begin{array}{c}
S (S Z) & S (S (S \bot)) \\
S Z & S (S \bot) \\
Z & S \bot \\
\bot \\
\end{array}
\]

Now consider the following data type declarations:

\[
\begin{align*}
\text{data U } &= \text{ V } \\
\text{data T a } &= \text{ C } | \text{ D (T a) } | \text{ E a a }
\end{align*}
\]

Give a graphical representation of the first three levels of the domain for the type T U. The third level contains the element D C, for example.
d) Consider the usual definitions for List a, i.e., \( \text{data List } a = \text{Nil} \mid \text{Cons } a (\text{List } a) \) and Nats from above.

Write a function \( \text{length :: List } a \rightarrow \text{Nats} \) in \textit{Simple Haskell} that computes the length of a list, i.e., \( \text{length (Cons } Z (\text{Cons } Z \text{Nil})) \) should yield \( S(S(Z)) \). Your solution should use the functions defined in the transformation from the lecture such as \( \text{sel}_{n,i} \), \( \text{isa} \), \( \text{argof} \), \( \text{constr} \), and \( \text{bot} \). You do not have to use the transformation rules from the lecture, though.
Exercise 4 (Lambda Calculus):  \(4 + 6 = 10\) points

a) Please translate the following Haskell expression into an equivalent lambda term (e.g., using \(\text{Lam}\)).

Translate the pre-defined function \(>\) to \(\text{GreaterThan}\), \(+\) to \(\text{Plus}\), \(*\) to \(\text{Times}\) and \(-\) to \(\text{Minus}\) (remember that the infix notation of \(>\), \(+\), \(*\), \(-\) is not allowed in lambda calculus). It suffices to give the result of the transformation:

\[
\text{let sqrt} = \lambda x \ a \rightarrow \text{if} \ a \ast a > x \text{then} \ a - 1 \text{else} \ \text{sqrt}\ x \ (a + 1) \text{ in} \ \text{sqrt u 0}
\]

b) Let \(t = \lambda x.\lambda y.\text{If} \ (\text{Eq} \ x \ y) \text{Nil} \ (\text{Cons} \ x \ \text{fromto} \ (\text{Plus} \ x \ 1) \ y)\) and

\[
\delta = \{ \begin{array}{l}
\text{If True} \rightarrow \lambda x.\lambda y.x, \\
\text{If False} \rightarrow \lambda x.\lambda y.y, \\
\text{Fix} \rightarrow \lambda f.\text{Fix} f
\end{array} \}
\cup \{ \text{Plus} \ x \ y \rightarrow z \mid x, y \in \mathbb{Z} \land z = x + y \}
\cup \{ \text{Eq} \ x \ y \rightarrow \text{False} \mid x, y \in \mathbb{Z} \land x \neq y \}
\cup \{ \text{Eq} \ x \ y \rightarrow \text{True} \mid x, y \in \mathbb{Z} \land x = y \}
\]

Please reduce \(\text{Fix} \ t \ 1 \ 2\) by WHNO-reduction with the \(\rightarrow_{\beta\delta}\)-relation. List all intermediate steps until reaching weak head normal form, but please write “\(t\)” instead of the term it represents whenever possible. However, you may combine several subsequent \(\rightarrow_{\beta}\)-steps.
Exercise 5 (Type Inference): (6 points)

Using the initial type assumption \( A_0 := \{ y :: \forall a. a \rightarrow a \} \) infer the type of the expression \( \lambda x.(y\,x)\,x \) using the algorithm \( W \).