Prof. Dr. J. Giesl F. Frohn

Exam in Functional Programming SS 2014 (V3B)

First Name: ______________________________

Last Name: ______________________________

Matriculation Number: ______________________________

Course of Studies (please mark **exactly** one):

- ○ Informatik Bachelor
- ○ Informatik Master
- ○ Other: ______________________
- ○ Mathematik Master
- ○ Software Systems Engineering Master

	Available Points	Achieved Points
Exercise 1	**9**	
Exercise 2	**19**	
Exercise 3	**11**	
Exercise 4	**29**	
Exercise 5	**12**	
Exercise 6	**10**	
Sum	**90**	

Notes:

- **On all sheets** (including additional sheets) you must write **your first name, your last name and your matriculation number**.
- Give your answers in readable and understandable form.
- Use **permanent** pens. Do not use red or green pens and do not use pencils.
- Please write your answers on the exam sheets (also use the reverse sides).
- For each exercise part, give **at most one** solution. Cancel out everything else. Otherwise all solutions of the particular exercise part will give you **0 points**.
- If we observe any **attempt of deception**, the whole exam will be evaluated to **0 points**.
- At the end of the exam, hand in **all sheets together with the sheets containing the exam questions**.

Exercise 1 (Quiz): (3 + 3 + 3 = 9 points)

a) Is `\f -> (f True) (f 1)` well typed in Haskell? Give the expression's type or briefly explain why it is not well typed.

b) Prove or disprove: If a relation $\succ \subseteq A \times A$ is confluent, then every element of A has a normal form with respect to $\succ$.

c) Are there monotonic functions which are not continuous? If so, give an example. Otherwise, give a brief explanation.

Exercise 2 (Programming in Haskell): (5 + 7 + 7 = 19 points)

We define a polymorphic data structure `Train` to represent trains that can contain different types of cargo.

```
data Train a
    = Locomotive (Train a)
    | Wagon a (Train a)
    | Empty deriving Show
```

The data structure `Cargo` is used to represent different types of cargo.

```
type Quantity = Int
type Weight = Int -- in kg
data Cargo
    = NoCargo
    | Persons Quantity
    | Goods Weight deriving Show
```

For example, `aTrain` is a valid expression of type `Train Cargo`.

```
aTrain = Locomotive (Wagon (Goods 100) (Wagon (Persons 10) (Wagon (Goods 200) Empty)))
```

Like `aTrain`, you can assume that every `Train` consists of a single `Locomotive` at its beginning followed by a sequence of `Wagons` and `Empty` at its end.
The following function can be used to *fold* a `Train`.

```
fold :: (a -> b -> b) -> b -> Train a -> b
fold _ res Empty = res
fold f res (Locomotive t) = fold f res t
fold f res (Wagon c t) = f c (fold f res t)
```

So for a `Train` `t`, `fold f res t` removes the constructor `Locomotive`, replaces `Wagon` by `f`, and replaces `Empty` by `res`.
In the following exercises, you are allowed to use predefined functions from the Haskell-Prelude.

a) Implement a function `filterTrain` together with its type declaration (`filterTrain :: ...`). The function `filterTrain` gets a predicate and an object of type `Train a` as input and returns an object of type `Train a` that only contains those wagons from the given `Train` whose cargo satisfies the predicate.

For example, assume that the function `areGoods` is implemented as follows:

```
areGoods :: Cargo -> Bool
areGoods (Goods _) = True
areGoods _ = False
```

Then the expression `filterTrain areGoods aTrain` should be evaluated to
`Locomotive (Wagon (Goods 100) (Wagon (Goods 200) Empty))`.

b) Implement a function `buildTrain :: [Cargo] -> Train Cargo`. In the resulting `Train`, a single `Wagon` must not contain more than 1000 kg of `Goods`. If the input list contains `Goods` that weigh more than 1000 kg, then these `Goods` must not be contained in the resulting train. Apart from this restriction, all the `Cargo` given via the input list has to be contained. Moreover, the resulting `Train` has to consist of a single `Locomotive` at its beginning, followed by a sequence of `Wagons` and `Empty` at its end. In your solution, you should use the function `filterTrain` even if you could not solve the previous exercise part.

For example, `buildTrain [Persons 10, Goods 2000, Goods 1000]` should be evaluated to the expression `Locomotive (Wagon (Persons 10) (Wagon (Goods 1000) Empty))`.

c) Implement a function `weight` together with its type declaration which computes the weight of all `Goods` in a train of type `Train Cargo`. For the definition of `weight`, use only one defining equation where the right-hand side is a call to the function `fold`.

For example, `weight aTrain` should be evaluated to `300`.

Exercise 3 (List Comprehensions): (3 + 3 + 5 = 11 points)

a) Write a Haskell function `divisors :: Int -> [Int]` to compute the list of all proper divisors of a given number `x`. Here, you can assume `x` ≥ 2. The result of `divisors x` includes 1, but not the number `x` itself. So for example, `divisors 6=[1,2,3]`. Use only one defining equation where the right-hand side is a list comprehension.

Hint: The function `mod :: Int -> Int -> Int` can be used to compute the modulo of two integers.

b) Write a Haskell expression in form of a list comprehension to compute all *perfect numbers*. A number x with $x \geq 2$ is perfect if and only if the sum of its proper divisors is equal to itself. For example, 6 is perfect, since its proper divisors are $1, 2$, and 3 and the sum of its proper divisors is 6. In your solution, you should use the function `divisors` even if you were not able to solve the previous exercise part.

Hint: The function `sum :: [Int] -> Int` computes the sum of a list of integers.

c) Write a Haskell expression in form of a list comprehension to compute all *semiperfect numbers*. A number x with $x \geq 2$ is semiperfect if and only if the sum of all or some of its proper divisors is equal to itself. For example, 12 is semiperfect: Its proper divisors are $1, 2, 3, 4$, and 6 and the sum of $2, 4$, and 6 is 12. In your solution, you should use the function `divisors` even if you were not able to solve exercise part (a). Moreover, you may use the function `sum` and the following functions:

- The function `exists :: (a -> Bool) -> [a] -> Bool` tests whether there is an element in the given list that satisfies the given predicate.
- The function `subsequences [a] -> [[a]]` computes all subsequences of the given list. For example, we have:

  ```
  subsequences [1,2,3] = [[],[1],[2],[1,2],[3],[1,3],[2,3],[1,2,3]]
  ```

Exercise 4 (Semantics): (10 + 10 + 6 + 3 = 29 points)

a) i) Let $L_{[]} = \{$`[]`,`[[]]`,`[[[]]]`$, \ldots\}$, i.e., $L_{[]}$ contains all lists where m opening brackets are followed by m closing brackets for an $m \in \mathbb{N} \setminus \{0\}$. Let $\leq_{nl} \subseteq L_{[]} \times L_{[]}$ be the relation that compares the *nesting-level* of two lists. More formally, if $nl(x)$ is the nesting level of the list x and $\leq \subset \mathbb{N} \times \mathbb{N}$ is the usual less-or-equal relation, then

$$l \leq_{nl} l' \iff nl(l) \leq nl(l')$$

So we have, e.g., `[]`$\leq_{nl}$`[[]]` because the nesting level of `[]` is one and the nesting level of `[[]]` is two.

1) Give an example for an infinite chain in $(L_{[]}, \leq_{nl})$.

2) Prove or disprove: the partial order $\leq_{nl}$ is complete on $L_{[]}$.

ii) Let L_0 be the set of all Haskell lists containing only zeros (so, e.g., $[] \in L_0$ and $[0, 0, 0] \in L_0$) and let $\leq_{len} \subseteq L_0 \times L_0$ be the relation that compares the *length* of two lists where all infinite lists are considered to have the same length. More formally, if $len(x)$ is the length of the list x and $\leq \subset \mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$ is the usual less-or-equal relation, then

$$l \leq_{len} l' \iff len(l) \leq len(l')$$

1) Give an example for an infinite chain in $(L_0, \leq_{len})$.

2) Prove or disprove: the partial order $\leq_{len}$ is complete on L_0.

b) i) Consider the following Haskell function `f`:

```
f :: (Int, Int) -> Int
f (x, 0) = 1
f (x, y) = x * f (x, y - 1)
```

Please give the Haskell declaration for the higher-order function `ff` corresponding to `f`, i.e., the higher-order function `ff` such that the least fixpoint of `ff` is `f`. In addition to the function declaration, please also give the type declaration for `ff`. You may use full Haskell for `ff`.

ii) Let $\phi_{\texttt{ff}}$ be the semantics of the function `ff`. Give the definition of $\phi_{\texttt{ff}}^{n}(\bot)$ in closed form for any $n \in \mathbb{N}$, i.e., give a non-recursive definition of the function that results from applying $\phi_{\texttt{ff}}$ n-times to $\bot$.

iii) Give the definition of the least fixpoint of $\phi_{\texttt{ff}}$ in closed form.

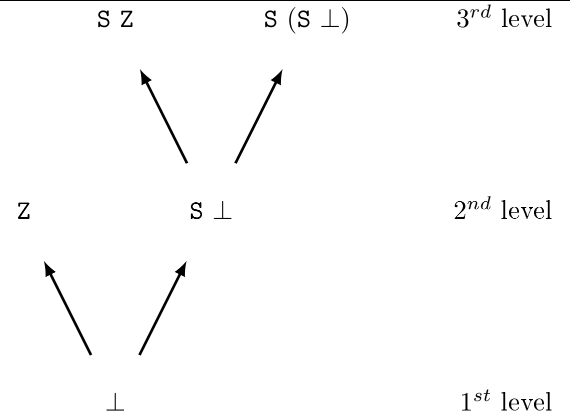
c) Consider the data type declarations on the left and, as an example, the graphical representation of the first three levels of the domain for `Nats` on the right:

```
data Nats = Z | S Nats

-------------------------------------

data Train a
    = L (Train a)    -- a Locomotive
    | W a (Train a) -- a Wagon
    | E              -- an Empty Train

data Cargo
    = NC     -- No Cargo
    | P Int -- n Persons
```



Give a graphical representation of the first three levels of the domain for the type `Train Cargo`. The third level contains the element `W (P ⊥) ⊥`, for example. Note that the domain for the type `Train Cargo` also contains `Trains` with multiple locomotives, `Trains` without `E` at their ends, and so on. In other words, the assumption from Exercise 2 ("Assume that every `Train` consists of a single `Locomotive` at its beginning followed by a sequence of `Wagons` and `Empty` at its end.") does *not* hold for this exercise.

d) Consider the definition for `Nats` from the previous exercise part, i.e., `data Nats = Z | S Nats`.

Moreover, consider the following Haskell function `f'`:

```
f' :: Int -> Int -> Int
f' x 0 = 1
f' x y = x * f' x (y - 1)
```

Write a function `fNat :: Nats -> Nats -> Nats` in **Simple** Haskell which, for natural numbers, computes the same result as the function `f'`. That means, if `n,m >= 0` and `f' n m = x`, then we have `fNat` $(S^n\ Z)\ (S^m\ Z) = S^x\ Z$. You can assume a predefined function `mult :: Nats -> Nats -> Nats` to multiply two natural numbers. However, there is no predefined function to subtract natural numbers of type `Nats`. Your solution should use the functions defined in the transformation from the lecture such as $\texttt{isa}_{\underline{\texttt{constr}}}$, $\texttt{argof}_{\underline{\texttt{constr}}}$, and `bot`. You do not have to use the transformation rules from the lecture, though.

Exercise 5 (Lambda Calculus): (4 + 8 = 12 points)

a) Reconsider the function `f'` from the previous exercise:

```
f' :: Int -> Int -> Int
f' x 0 = 1
f' x y = x * f' x (y - 1)
```

Please implement this function in the Lambda Calculus, i.e., give a term f such that, for all $\mathtt{x}, \mathtt{y}, \mathtt{z} \in \mathbb{Z}$, `f' x y == z` if and only if $f\,\mathtt{x}\,\mathtt{y}$ can be reduced to `z` via WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation and the set of rules δ as introduced in the lecture to implement Haskell. You can use infix notation for predefined functions like $(==)$, $(*)$ or $(-)$.

b) Let

$$t = \lambda\, add\ x\ y.\, \texttt{if}\ (y == 0)\ x\ (add\ (x+1)\ (y-1))$$

and

$$\begin{aligned}\delta = \ & \{\ \texttt{if True} \rightarrow \lambda\, x\ y.\, x,\\ & \quad \texttt{if False} \rightarrow \lambda\, x\ y.\, y,\\ & \quad \texttt{fix} \rightarrow \lambda\, f.\, f(\texttt{fix}\ f)\}\\ \cup & \ \{\ x - y \rightarrow z \mid x, y, z \in \mathbb{Z} \wedge z = x - y\}\\ \cup & \ \{\ x + y \rightarrow z \mid x, y, z \in \mathbb{Z} \wedge z = x + y\}\\ \cup & \ \{\ x == x \rightarrow \texttt{True} \mid x \in \mathbb{Z}\}\\ \cup & \ \{\ x == y \rightarrow \texttt{False} \mid x, y \in \mathbb{Z}, x \neq y\}\end{aligned}$$

Please reduce $\texttt{fix}\, t\, 0\, 0$ by WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation. List **all** intermediate steps until reaching weak head normal form, but please write "t" instead of

$$\lambda\, add\ x\ y.\, \texttt{if}\ (y == 0)\ x\ (add\ (x+1)\ (y-1))$$

whenever possible.

Exercise 6 (Type Inference): (10 points)

Using the initial type assumption $A_0 := \{x :: \forall a.a\}$, infer the type of the expression $\lambda f.f\,(f\,x)$ using the algorithm $\mathcal{W}$.