Prof. Dr. J. Giesl F. Frohn

Exam in Functional Programming SS 2014 (V3B)

First Name: ______________________________

Last Name: ______________________________

Matriculation Number: ______________________________

Course of Studies (please mark **exactly** one):

- ○ Informatik Bachelor
- ○ Informatik Master
- ○ Other: ______________________________
- ○ Mathematik Master
- ○ Software Systems Engineering Master

	Available Points	Achieved Points
Exercise 1	**9**	
Exercise 2	**20**	
Exercise 3	**11**	
Exercise 4	**28**	
Exercise 5	**12**	
Exercise 6	**10**	
Sum	**90**	

Notes:

- **On all sheets** (including additional sheets) you must write **your first name, your last name and your matriculation number**.
- Give your answers in readable and understandable form.
- Use **permanent** pens. Do not use red or green pens and do not use pencils.
- Please write your answers on the exam sheets (also use the reverse sides).
- For each exercise part, give **at most one** solution. Cancel out everything else. Otherwise all solutions of the particular exercise part will give you **0 points**.
- If we observe any **attempt of deception**, the whole exam will be evaluated to **0 points**.
- At the end of the exam, hand in **all sheets together with the sheets containing the exam questions**.

Exercise 1 (Quiz): (3 + 3 + 3 = 9 points)

a) Give a type declaration for `f` such that `(f True) (f 1)` is well typed in Haskell or explain why such a type declaration cannot exist.

b) Prove or disprove: If $\succ \subseteq A \times A$ is confluent, then each $a \in A$ has at most one normal form w.r.t. $\succ$.

c) What is the connection between monotonicity, continuity, and computability?

Exercise 2 (Programming in Haskell): (6 + 7 + 7 = 20 points)

We define a polymorphic data structure `Tree e` for binary trees whose nodes store values of type `e`.

```
data Tree e = Node e (Tree e) (Tree e) | Empty
```

The data structure `Forest e` is used to represent lists of trees.

```
type Forest e = [Tree e]
```

Furthermore, we define the following data structure:

```
data Animal = Squirrel | None
```

For example, `aForest` is a valid expression of type `Forest Animal`.

```
aForest = [Node Squirrel Empty (Node Squirrel Empty Empty), Node None Empty Empty]
```

In this exercise, you may use full Haskell and predefined functions from the Haskell Prelude.

a) Implement a function `hunt` together with its type declaration that removes all `Squirrel`s from a `Forest Animal`, i.e., each occurrence of a `Squirrel` should be replaced by `None`.

For example, `hunt aForest` should be evaluated to `[Node None Empty (Node None Empty Empty), Node None Empty Empty]`.

b) Implement a function `fold :: (e -> res -> res -> res) -> res -> Tree e -> res` to fold a `Tree`. The first argument of `fold` is the function that is used to combine the value of the current `Node` with the subresults obtained for the two direct subtrees of the current `Node`. The second argument of `fold` is the start value, i.e., the initial subresult. The third argument is the `Tree` that has to be folded. So for a `Tree t`, `fold f x t` replaces the constructor `Node` by `f` and the constructor `Empty` by `x`.

As an example, consider the following function:

```
count :: Animal -> Int -> Int -> Int
count Squirrel x y = x + y + 1
count None x y = x + y
```

Then `fold count 0 (Node Squirrel Empty (Node Squirrel Empty Empty))` should evaluate to `2`, i.e., this application of `fold` counts all `Squirrel`s in a `Tree`.

c) Implement a function `isInhabited` together with its type declaration which gets a `Forest Animal` as input and returns `True` if and only if there is a `Tree` in the `Forest` that contains a `Squirrel`. For the definition of `isInhabited`, use only one defining equation where the right-hand side contains a call to the function `fold`. Of course, you may (and have to) use the function `fold` even if you were not able to solve exercise part (b). Moreover, you may use the function `count` from exercise part (b).

Note that the function `fold` operates on a `Tree`, whereas the function `isInhabited` operates on a `Forest`!

Exercise 3 (List Comprehensions): (4 + 7 = 11 points)

In this exercise, you can assume that there exists a function `divisors :: Int -> [Int]` where, for any natural number `x` ≥ 2, `divisors x` computes the list of all its proper divisors (including 1, but excluding `x`). So for example, `divisors 6 = [1,2,3]`.

a) Write a Haskell expression in form of a list comprehension to compute all *amicable pairs of numbers*. A pair of natural numbers (x, y) with $x > y \geq 2$ is amicable if and only if the sum of the proper divisors of x is equal to y and the sum of the proper divisors of y is equal to x. For example, $(284, 220)$ is amicable:

- The proper divisors of 284 are $1, 2, 4, 71$, and 142 and their sum is 220.
- The proper divisors of 220 are $1, 2, 4, 5, 10, 11, 20, 22, 44, 55$, and 110 and their sum is 284.

In other words, give a list comprehension for a list that only contains amicable pairs of numbers and, for every amicable pair of numbers p, there is an $n \in \mathbb{N}$ such that the n^{th} element of the list is p.

Hint: The function `sum :: [Int] -> Int` computes the sum of a list of integers.

b) Write a Haskell expression in form of a list comprehension to compute all *practical numbers*. A natural number $x \geq 2$ is practical if and only if each smaller number $y \in \{1, \ldots, x - 1\}$ is equal to the sum of some of x's proper divisors. For example, 6 is practical: Its proper divisors are $1, 2$, and 3 and we have $4 = 3 + 1$ and $5 = 3 + 2$.

In your solution, you may use the function `sum` and the following functions:

- The function `any :: (a -> Bool) -> [a] -> Bool` tests whether there is an element in the given list that satisfies the given predicate.
- The function `all :: (a -> Bool) -> [a] -> Bool` tests whether all elements in the given list satisfy the given predicate.
- The function `subsequences [a] -> [[a]]` computes all subsequences of the given list. For example, we have:

  `subsequences [1,2,3] = [[],[1],[2],[1,2],[3],[1,3],[2,3],[1,2,3]]`

Exercise 4 (Semantics): (12 + 7 + 5 + 4 = 28 points)

a) i) Let $\mathbb{N}^\infty$ be the set of all infinite sequences of natural numbers (e.g., $[0, 0, 2, 2, 4, 4, \dots] \in \mathbb{N}^\infty$) and let $\leq_p \subseteq \mathbb{N}^\infty \times \mathbb{N}^\infty$ be the relation that compares infinite sequences of natural numbers by their *prefix sums*. The n^{th} prefix sum $p_n(s)$ for some $n \in \mathbb{N}$ of a sequence $s \in \mathbb{N}^\infty$ is the sum of the first n elements of s. We have $s \leq_p s'$ if and only if $s = s'$ or there is an $n \in \mathbb{N}$ such that $p_n(s) < p_n(s')$ and $p_m(s) = p_m(s')$ for all $m \in \{0, \dots, n-1\}$.

1) Prove that $\leq_p$ is transitive.

2) Give an example for an infinite chain in $(\mathbb{N}^\infty, \leq_p)$.

3) Prove or disprove: The partial order $\leq_p$ is complete on $\mathbb{N}^\infty$.

ii) Prove or disprove: The partial order $\leq$ is complete on $\mathbb{N}$. Here, $\leq$ is the usual "less than or equal" relation.

b) i) Consider the following Haskell function `f`:

```
f :: Int -> Int
f 0 = 1
f x = x * x * f (x - 1)
```

Please give the Haskell declaration for the higher-order function `ff` corresponding to `f`, i.e., the higher-order function `ff` such that the least fixpoint of `ff` is `f`. In addition to the function declaration, please also give the type declaration of `ff`. You may use full Haskell for `ff`.

ii) Let $\phi_{\texttt{ff}}$ be the semantics of the function `ff`. Give the least fixpoint of $\phi_{\texttt{ff}}$ in closed form, i.e., give a non-recursive definition of the least fixpoint of $\phi_{\texttt{ff}}$.

Hint: For natural numbers x, the factorial function can be defined as follows:

$$x! = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot (x-1)! & \text{if } x > 0 \end{cases}$$

c) Consider the data type declarations on the left and, as an example, the graphical representation of the first three levels of the domain for `Nats` on the right:

```
data Nats = Z | S Nats

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type Forest e = [Tree e]
data Tree e =
    Node e (Tree e) (Tree e)
    | Empty
```

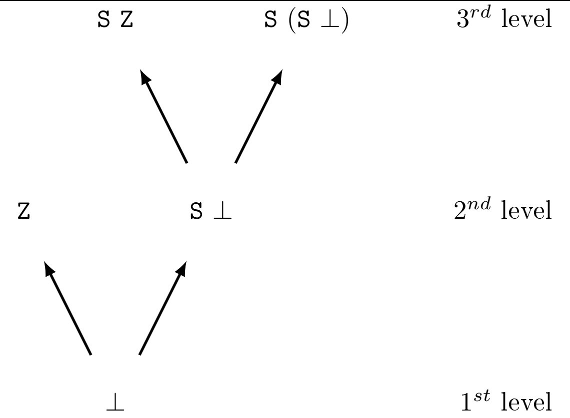

Give a graphical representation of the first three levels of the domain for the type `Forest Int`. The third level contains the element `Empty:`⊥, for example.

d) Reconsider the definition for `Nats` from the previous exercise part, i.e., `data Nats = Z | S Nats`. Moreover, reconsider the function `f`:

```
f :: Int -> Int
f 0 = 1
f x = x * x * f (x - 1)
```

Write a function `fNat :: Nats -> Nats` in **Simple** Haskell which, for natural numbers, computes the same result as the function `f`. That means, if `n >= 0` and `f n = x`, then we have `fNat` $(\texttt{S}^{\texttt{n}}\ \texttt{Z}) = \texttt{S}^{\texttt{x}}\ \texttt{Z}$. You can assume that there exists a predefined function `mult :: Nats -> Nats -> Nats` to multiply two natural numbers. However, there is no predefined function to subtract natural numbers of type `Nats`. Your solution should use the functions defined in the transformation from the lecture such as $\texttt{isa}_{\underline{\texttt{constr}}}$ and $\texttt{argof}_{\underline{\texttt{constr}}}$. You do not have to use the transformation rules from the lecture, though.

Exercise 5 (Lambda Calculus): (4 + 8 = 12 points)

a) Reconsider the function `f` from the previous exercise:

```
f :: Int -> Int
f 0 = 1
f x = x * x * f (x - 1)
```

Please implement this function in the Lambda Calculus, i.e., give a term `t` such that, for all `x`, `y` $\in \mathbb{Z}$, `f x == y` if and only if `t x` can be reduced to `y` via WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation and the set of rules δ as introduced in the lecture to implement Haskell. You can use infix notation for predefined functions like $(==)$, $(*)$ or $(-)$.

b) Let $\texttt{t} = \lambda g\ x.\texttt{if}\ (x == 0)\ x\ (g\ x)$ and

$$\begin{aligned}\delta = &\ \{\ \texttt{if True} \rightarrow \lambda\, x\ y.\, x,\\ &\quad \texttt{if False} \rightarrow \lambda\, x\ y.\, y,\\ &\quad \texttt{fix} \rightarrow \lambda\, f.\, f(\texttt{fix}\ f)\}\\ &\cup\ \{\ x == x \rightarrow \texttt{True} \mid x \in \mathbb{Z}\}\\ &\cup\ \{\ x == y \rightarrow \texttt{False} \mid x, y \in \mathbb{Z}, x \neq y\}\end{aligned}$$

Please reduce $\texttt{fix}\,\texttt{t}\,0$ by WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation. List **all** intermediate steps until reaching weak head normal form, but please write "`t`" instead of $\lambda g\ x.\texttt{if}\ (x == 0)\ x\ (g\ x)$ whenever possible.

Exercise 6 (Type Inference): (10 points)

Using the initial type assumption $A_0 := \{x :: \forall a.a, g :: \forall a.a\}$, infer the type of the expression $\lambda f.g\,(f\,x)$ using the algorithm $\mathcal{W}$.