2.1.2 Monotonic and Continuous Functions (Part 2)

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For the function domain, we only regard monotonic functions (all computable functions are monotonic).
Now we introduce another restriction: continuous functions (stetig)

(Slide 32)

$$\begin{array}{ccccccccc} d_1 & \sqsubseteq & d_2 & \sqsubseteq & d_3 & \sqsubseteq & \dots & \xrightarrow{lub} & d \\ \downarrow f & & \downarrow f & & \downarrow f & & & & \downarrow f \\ f(d_1) & \sqsubseteq & f(d_2) & \sqsubseteq & f(d_3) & \sqsubseteq & \dots & \xrightarrow{lub} & f(d) \end{array}$$

If the "black path" and the "red path" end in the same result, then we say that f is continuous.

Def. 2.1.14 (Continuity)

Let $\sqsubseteq_{D_1}$ and $\sqsubseteq_{D_2}$ be cpo's on D_1 and D_2, resp.
A function $f: D_1 \to D_2$ is continuous iff for all chains S of D_1, we have $f(\sqcup S) = \sqcup \{f(d) \mid d \in S\}$.
Let $\langle D_1 \to D_2 \rangle$ denote the set of all continuous functions from D_1 to D_2.

Literature: $[D_1 \to D_2]$, but could be confused with Haskell's list notation.

Sometimes, one writes $f(S)$ for $\{f(d) \mid d \in S\}$.
Then continuity of f means $f(\sqcup S) = \sqcup f(S)$

Examples to illustrate continuity

1. Constant functions are always continuous

$f : D_1 \to D_2$ with $f(x) = e$ for some $e \in D_2$.

f is continuous: if S is a chain of D_1, then

$$f(S) = \{ f(d) \mid d \in S \} = \{e\}$$

$$\bigsqcup f(S) = e \qquad\qquad f(\bigsqcup S) = e$$

2. The identity function is also continuous

$i : D \to D$ where $i(x) = x$ for all $x \in D$.

$$\bigsqcup i(S) = \bigsqcup S \qquad\qquad i(\bigsqcup S) = \bigsqcup S$$

3. The following function is not continuous:

$g : (\mathbb{Z}_\perp \to \mathbb{Z}_\perp) \to \mathbb{Z}_\perp$ where

$$g(f) = \begin{cases} 0, & \text{if } f(x) \neq \perp_{\mathbb{Z}_\perp} \text{ for all } x \in \mathbb{Z} \\ \perp_{\mathbb{Z}_\perp}, & \text{otherwise} \end{cases}$$

g is a semi-decision for totality: if f is total, then g terminates with "0". Otherwise, g does not terminate.

However: totality is not semi-decidable

$\leadsto$ g is not computable

g is monotonic: if $f_1 \sqsubseteq f_2$ then:

Case 1: $f_1(x) = f_2(x)$ for all $x \in \mathbb{Z}$

$\leadsto g(f_1) = g(f_2)$

Case 2: there is an $x \in \mathbb{Z}$ where $f_1(x) \neq f_2(x)$

$\leadsto f_1(x) = \bot$

$\leadsto g(f_1) = \bot \sqsubseteq g(f_2)$.

g is not continuous:

We consider the chain $S = \{ fact_0, fact_1, \dots \}$

All $fact_n$ are not total ($fact_n(x) = \bot$ for $n \leq x$).

$g(fact_0) = g(fact_1) = \dots = \bot$.

$\bigsqcup g(S) = \bigsqcup \{\bot\} = \bot$. $\qquad g(\underbrace{\bigsqcup S}_{fact}) = 0$

different!

A chain of partial functions can have a lub that is total.

The example above shows: monotonicity $\not\leadsto$ continuity.

But the converse holds: continuity $\leadsto$ monotonicity.

Thm 7.1.15 (Continuity + Monotonicity)

Let $\sqsubseteq_{D_1}$, $\sqsubseteq_{D_2}$ be cpo's on D_1 and D_2, resp., let $f: D_1 \to D_2$.

(a) f is continuous iff f is monotonic and $f(\sqcup S) \sqsubseteq \sqcup f(S)$.

(b) If D_1 has only finite chains, then f is continuous iff f is monotonic.

<u>Proof</u>: (a) "$\Rightarrow$": Let f be continuous. Then we even have $f(\sqcup S) = \sqcup f(S)$.

To show: f is monotonic. Let $d, d' \in D_1$ with $d \sqsubseteq d'$.

Then $S = \{d, d'\}$ is a chain.

Thus: $f(d') = f(\underbrace{\sqcup S}_{d'}) \underset{\substack{\text{Since } f \\ \text{is continuous}}}{=} \sqcup f(S) = \sqcup \{f(d), f(d')\}$

This implies $f(d) \sqsubseteq f(d')$.

"$\Leftarrow$": Let f be monotonic and $f(\sqcup S) \sqsubseteq \sqcup f(S)$.

To show: $\sqcup f(S) \sqsubseteq f(\sqcup S)$.

It suffices to show that $f(\sqcup S)$ is an upper bound of $f(S)$.

Let $e \in f(S)$, i.e., there is a $d \in S$ with $f(d) = e$.

Since $d \sqsubseteq \sqcup S \underset{\substack{f \text{ is} \\ \text{monotonic}}}{\curvearrowright} e = f(d) \sqsubseteq f(\sqcup S)$

(b) Due to (a), we only have to show that for every finite chain $S = \{d_1, \dots, d_n\}$ with $d_1 \sqsubseteq d_2 \sqsubseteq \dots \sqsubseteq d_n$ and any monotonic f, we have $f(\underbrace{\sqcup S}_{d_n}) \sqsubseteq \sqcup f(S)$.

We have: $f(\sqcup S) = f(d_n) \in f(S)$

Thus: $f(\sqcup S) \sqsubseteq \sqcup f(S)$.

$\boxed{\text{⚡}}$

The following theorem states that if we restrict ourselves to continuous functions, then still every chain of (cont.) functions has a lub which is also continuous. Thus: $\sqsubseteq$ is also complete on $\langle D_1 \to D_2 \rangle$.

<u>Thm 2.1.16</u> (Completeness of Function Domain)
Let D_1, D_2 be domains with corresponding cpo's. Then $\langle D_1 \to D_2 \rangle$ is the corresponding function domain and $\sqsubseteq_{D_1 \to D_2}$ is a cpo on $\langle D_1 \to D_2 \rangle$.