Transforming Context-Sensitive Rewrite Systems*

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Abstract. We present two new transformation techniques for proving termination of context-sensitive rewriting. Our first method is simple, sound, and more powerful than previously suggested transformations. However, it is not complete, i.e., there are terminating context-sensitive rewrite systems that are transformed into non-terminating term rewrite systems. The second method that we present in this paper is both sound and complete. This latter result can be interpreted as stating that from a termination perspective there is no reason to study context-sensitive rewriting.

1 Introduction

In the presence of infinite reductions in term rewriting, the search for normal forms is usually guided by adopting a suitable reduction strategy. Consider the following rewrite rules which form a part of a term rewrite system that implements the Sieve of Eratosthenes for generating the infinite list of all prime numbers (we did not include the rules defining $\mathsf{divides}$):

$$\begin{array}{lll} \mathsf{primes} & \to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) & \mathsf{head}(x : y) \to x \\ \mathsf{from}(x) & \to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{tail}(x : y) \;\; \to y \\ \mathsf{if}(\mathsf{true}, x, y) & \to x & \mathsf{sieve}(x : y) \to x : \mathsf{filter}(x, \mathsf{sieve}(y)) \\ \mathsf{if}(\mathsf{false}, x, y) & \to y & \\ \mathsf{filter}(\mathsf{s}(\mathsf{s}(x)), y : z) & \multicolumn{2}{l}{\to \mathsf{if}(\mathsf{divides}(\mathsf{s}(\mathsf{s}(x)), y),\, \mathsf{filter}(\mathsf{s}(\mathsf{s}(x)), z),\, y : \mathsf{filter}(\mathsf{s}(\mathsf{s}(x)), z))} \end{array}$$

A term like $\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes})))$ admits a finite reduction to the normal form $\mathsf{s}^5(\mathsf{0})$ (the third prime number) as well as infinite reductions. The infinite reductions can for instance be avoided by always contracting the leftmost-outermost redex. Context-sensitive rewriting (Lucas [10, 11]) provides an alternative way

* Appeared in the *Proceedings of the 10th International Conference on Rewriting Techniques and Applications (RTA-99)*, Trento, Italy, Lecture Notes in Computer Science 1631, pp. 271-285, Springer-Verlag, 1999.

of solving the non-termination problem. Rather than specifying which redexes may be contracted, in context-sensitive rewriting for every function symbol one indicates which arguments may not be evaluated and a contraction of a redex is allowed only if it is does not take place in a forbidden argument of a function symbol above it. For instance, by forbidding all contractions in the argument t of a term of the form $s : t$, infinite reductions are no longer possible while normal forms can still be computed. This example illustrates that this restricted form of rewriting has strong connections with lazy evaluation strategies used in functional programming languages, because it allows us to deal with non-terminating programs and infinite data structures, cf. [11].

In this paper we are concerned with the problem of showing termination of context-sensitive rewriting. More precisely, we consider transformations from context-sensitive rewrite systems to ordinary term rewrite systems that are *sound* with respect to termination: termination of the transformed term rewrite system implies termination of the original context-sensitive rewrite system. The advantage of such an approach is that all techniques for proving termination of term rewriting (e.g., [3, 6, 8, 14]) can be used to infer termination of context-sensitive rewriting. Two such transformations are reported in the literature, by Lucas [10] and by Zantema [17]. We add two more. Our first transformation is simple, its soundness is easily established, and it improves upon the transformations of [10, 17]. To be precise, we prove that the class of terminating context-sensitive rewrite systems for which our transformation succeeds is larger than that of Lucas' transformation and we claim that the same holds for Zantema's transformation. None of these three transformations succeeds in transforming every terminating context-sensitive rewrite system into a terminating term rewrite system. In other words, they all lack *completeness*. We analyze the failure of completeness for our first transformation, resulting in a second transformation with is both sound and complete. Hence it appears that from a termination point of view there is no reason to study context-sensitive rewriting further. We come back to this issue in the final part of the paper.

The remainder of the paper is organized as follows. In the next section we recall the definition of context-sensitive rewriting as well as the previous transformations of Lucas and Zantema. In Section 3 we present our first transformation and prove that it is sound. Despite being incomplete, we argue that it can handle more systems than the transformations of Lucas and Zantema. In Section 4 we refine our first transformation into a sound and complete one. The bulk of this section is devoted to the completeness proof. We make some concluding remarks in Section 5.

2 Preliminaries and Related Work

Familiarity with the basics of term rewriting ([4, 7, 9]) is assumed. Let $\mathcal{F}$ be a signature. A function $\mu\colon \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is called a *replacement map* if $1 \leqslant i \leqslant \text{arity}(f)$ for all $f \in \mathcal{F}$ and $i \in \mu(f)$. A *context-sensitive rewrite system* (CSRS for short) is a term rewrite system (TRS) $\mathcal{R}$ over a signature $\mathcal{F}$ that is equipped with a replacement map μ. We always assume that $\mathcal{F}$ contains a constant. The context-sensitive rewrite relation $\rightarrow_{\mathcal{R},\mu}$ is defined as the restriction of the usual rewrite relation $\rightarrow_{\mathcal{R}}$ to contractions of redexes at *active* positions. A position π in a term t is (μ-)active if $\pi = \varepsilon$ (the root position), or $t = f(t_1, \ldots, t_n)$, $\pi = i{\cdot}\pi'$, $i \in \mu(f)$, and π' is active in t_i. So $s \rightarrow_{\mathcal{R},\mu} t$ if and only if there exist a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$.

Consider the TRS of the introduction. By taking $\mu(:) = \mu(\mathsf{if}) = \mu(\mathsf{sieve}) = \mu(\mathsf{from}) = \mu(\mathsf{s}) = \mu(\mathsf{head}) = \mu(\mathsf{tail}) = \{1\}$, and $\mu(\mathsf{filter}) = \mu(\mathsf{divides}) = \{1, 2\}$ we obtain a terminating CSRS. The term $\mathsf{0} : \mathsf{from}(\mathsf{s}(\mathsf{0}))$, which has an infinite reduction in the TRS, is a normal form of the CSRS because the reduction step to $\mathsf{0} : (\mathsf{s}(\mathsf{0}) : \mathsf{from}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$ is no longer possible as the contracted redex occurs at a forbidden position ($2 \notin \mu(:)$).

Context-sensitive rewriting subsumes ordinary rewriting (by taking $\mu(f) = \{1, ..., n\}$ for every n-ary function symbol f). The interesting case is when $\mathcal{R}$ admits infinite reductions and μ is defined in such a way that $\rightarrow_{\mathcal{R},\mu}$ is terminating but still capable of computing ($\mathcal{R}$-)normal forms. For the latter aspect we refer to Lucas [11]; in this paper we are only concerned with termination of context-sensitive rewriting.

Lucas [10] presented a simple transformation from CSRSs to TRSs which is sound with respect to termination. Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The idea of the transformation is to replace every function symbol $f \in \mathcal{F}$ by a new function symbol f_μ where all arguments except the active ones are removed. Thus, the arity of f_μ is $|\mu(f)|$. The transformed system $\mathcal{R}^{\mathrm{L}}_\mu$ results from $\mathcal{R}$ by normalising all terms in its rewrite rules using the (terminating and confluent) TRS consisting of all rules

$$f(x_1, \ldots, x_n) \rightarrow f_\mu(x_{i_1}, \ldots, x_{i_k})$$

such that $\mu(f) = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$. For instance, if $\mathcal{R}$ is the TRS of the introduction and μ is defined as above, then $\mathcal{R}^{\mathrm{L}}_\mu$ consists of the following rewrite rules:

$$\begin{array}{lll}
\mathsf{primes}_\mu & \rightarrow \mathsf{sieve}_\mu(\mathsf{from}_\mu(\mathsf{s}_\mu(\mathsf{s}_\mu(\mathsf{0}_\mu)))) & \mathsf{head}_\mu(:_\mu(x)) \rightarrow x \\
\mathsf{from}_\mu(x) & \rightarrow :_\mu(x) & \mathsf{tail}_\mu(:_\mu(x)) \;\; \rightarrow y \\
\mathsf{sieve}_\mu(:_\mu(x)) & \rightarrow :_\mu(x) & \\
\mathsf{filter}_\mu(\mathsf{s}_\mu(\mathsf{s}_\mu(x)), :_\mu(y)) & \rightarrow \mathsf{if}_\mu(\mathsf{divides}_\mu(\mathsf{s}_\mu(\mathsf{s}_\mu(x)), y)) & \\
\mathsf{if}_\mu(\mathsf{true}_\mu) & \rightarrow x & \\
\mathsf{if}_\mu(\mathsf{false}_\mu) & \rightarrow y &
\end{array}$$

Note that $\mathcal{R}^{\mathrm{L}}_\mu$ is not terminating due to the extra variables in the right-hand sides of the rules for tail_μ and if_μ.

Zantema [17] presented a more complicated transformation in which subterms at forbidden positions are marked rather than discarded. The transformed system $\mathcal{R}^{\mathrm{Z}}_\mu$ consists of two parts. The first part results from a translation of the rewrite rules of $\mathcal{R}$, as follows. Every function symbol f occurring in a left or right-hand side is replaced by $\underline{f}$ (a fresh function symbol of the same arity as

f) if it occurs in a forbidden argument of the function symbol directly above it. These new function symbols are used to block further reductions at this position.

In addition, if a variable x occurs in a forbidden position in the left-hand side l of a rewrite rule $l \to r$ then all occurrences of x in r are replaced by $\mathsf{a}(x)$. Here a is a new unary function symbol which is used to activate blocked function symbols again. The second part of $\mathcal{R}_\mu^{\mathrm{Z}}$ consists of rewrite rules that are needed for blocking and unblocking function symbols:

$$\begin{aligned} f(x_1,\dots,x_n) &\to \underline{f}(x_1,\dots,x_n) \\ \mathsf{a}(\underline{f}(x_1,\dots,x_n)) &\to f(x_1,\dots,x_n) \end{aligned}$$

for every n-ary f for which $\underline{f}$ appears in the first part of $\mathcal{R}_\mu^{\mathrm{Z}}$, together with the rule $\mathsf{a}(x) \to x$. The example CSRS $(\mathcal{R},\mu)$ is transformed into

$$\begin{array}{ll} \mathsf{primes} & \to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) \\ \mathsf{from}(x) & \to x : \underline{\mathsf{from}}(\mathsf{s}(x)) \\ \mathsf{sieve}(x:y) & \to x : \underline{\mathsf{filter}}(x, \mathsf{sieve}(\mathsf{a}(y))) \\ \mathsf{filter}(\mathsf{s}(\mathsf{s}(x)), y:z) & \to \mathsf{if}(\mathsf{divides}(\mathsf{s}(\mathsf{s}(x)),y), \underline{\mathsf{filter}}(\mathsf{s}(\mathsf{s}(x)),\mathsf{a}(z)), \\ & \qquad\qquad\qquad y \mathbin{\underline{:}} \underline{\mathsf{filter}}(\mathsf{s}(\mathsf{s}(x)),\mathsf{a}(z))) \end{array}$$

$$\begin{array}{llll} \mathsf{if}(\mathsf{true},x,y) & \to \mathsf{a}(x) & & \\ \mathsf{if}(\mathsf{false},x,y) & \to \mathsf{a}(y) & \mathsf{head}(x:y) & \to x \\ \mathsf{from}(x) & \to \underline{\mathsf{from}}(x) & \mathsf{tail}(x:y) & \to \mathsf{a}(y) \\ \mathsf{sieve}(x) & \to \underline{\mathsf{sieve}}(x) & \mathsf{a}(\underline{\mathsf{from}}(x)) & \to \mathsf{from}(x) \\ \mathsf{filter}(x,y) & \to \underline{\mathsf{filter}}(x,y) & \mathsf{a}(\underline{\mathsf{sieve}}(x)) & \to \mathsf{sieve}(x) \\ x:y & \to x \mathbin{\underline{:}} y & \mathsf{a}(\underline{\mathsf{filter}}(x,y)) & \to \mathsf{filter}(x,y) \\ \mathsf{a}(x) & \to x & \mathsf{a}(x \mathbin{\underline{:}} y) & \to x:y \end{array}$$

This transformation is sound but not complete as we have the infinite reduction

$$\begin{aligned} \mathsf{sieve}(\mathsf{a}(\underline{\mathsf{from}}(\mathsf{0}))) &\to^+_{\mathcal{R}_\mu^{\mathrm{Z}}} \mathsf{0} : \underline{\mathsf{filter}}(\mathsf{0}, \mathsf{sieve}(\mathsf{a}(\underline{\mathsf{from}}(\mathsf{s}(\mathsf{0}))))) \\ &\to^+_{\mathcal{R}_\mu^{\mathrm{Z}}} \mathsf{0} : \underline{\mathsf{filter}}(\mathsf{0}, \mathsf{s}(\mathsf{0}) : \underline{\mathsf{filter}}(\mathsf{s}(\mathsf{0}), \mathsf{sieve}(\mathsf{a}(\underline{\mathsf{from}}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))))) \\ &\to^+_{\mathcal{R}_\mu^{\mathrm{Z}}} \cdots \end{aligned}$$

in the TRS $\mathcal{R}_\mu^{\mathrm{Z}}$.

Zantema's method appears to be more powerful than Lucas' transformation but actually the two methods are incomparable (cf. the TRS consisting of the single rule $\mathsf{c} \to \mathsf{f}(\mathsf{g}(\mathsf{c}))$ with $\mu(\mathsf{f}) = \varnothing$ and $\mu(\mathsf{g}) = \{1\}$).

3 A Sound Transformation

In this section we present our first transformation from CSRSs to TRSs. The advantage of this transformation is that it is very easy and more powerful than the transformations of Lucas and Zantema defined in the preceding section. In the transformation we will extend the original signature $\mathcal{F}$ of the TRS by two additional unary function symbols active and mark.

Essentially, the idea for the transformation is to mark the active positions in a term on the object level, because those positions are the only ones where context-sensitive rewriting may take place. For this purpose we use the new function symbol active. Thus, instead of a rule $l \to r$ the transformed TRS should contain a rule whose left-hand side is $\mathsf{active}(l)$. Moreover, after rewriting an instance of l to the corresponding instance of r, we have to mark the new active positions in the resulting term. For that purpose we use the function mark. So we replace every rule $l \to r$ by $\mathsf{active}(l) \to \mathsf{mark}(r)$. To mark all active positions in a term, the rules for mark must have the form

$$\mathsf{mark}(f(x_1, \ldots, x_n)) \to \mathsf{active}(f([x_1], \ldots, [x_n]))$$

where the form of the argument $[x_i]$ depends on whether i is an active argument of f: If $i \in \mu(f)$ then x_i must also be marked active and thus $[x_i] = \mathsf{mark}(x_i)$, otherwise the ith argument of f is not active and we define $[x_i] = x_i$. Finally, we also need a rule to deactivate terms. For example, consider the TRS consisting of the following rewrite rules:

$$\begin{aligned} \mathsf{a} &\to \mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{a} \\ \mathsf{b} &\to \mathsf{c} \end{aligned}$$

No matter how the replacement map μ is defined, the resulting CSRS is not terminating. Suppose $\mu(f) = \{1\}$. In the transformed system we would have the rules

$$\begin{array}{llll} \mathsf{active}(\mathsf{a}) & \to \mathsf{mark}(\mathsf{f}(\mathsf{b})) & \mathsf{mark}(\mathsf{a}) & \to \mathsf{active}(\mathsf{a}) \\ \mathsf{active}(\mathsf{f}(\mathsf{b})) & \to \mathsf{mark}(\mathsf{a}) & \mathsf{mark}(\mathsf{b}) & \to \mathsf{active}(\mathsf{b}) \\ \mathsf{active}(\mathsf{b}) & \to \mathsf{mark}(\mathsf{c}) & \mathsf{mark}(\mathsf{c}) & \to \mathsf{active}(\mathsf{c}) \\ & & \mathsf{mark}(\mathsf{f}(x)) & \to \mathsf{active}(\mathsf{f}(\mathsf{mark}(x))) \end{array}$$

This TRS is terminating because $\mathsf{active}(\mathsf{a})$ can be reduced to $\mathsf{active}(\mathsf{f}(\mathsf{active}(\mathsf{b})))$, but if we cannot deactivate the subterm $\mathsf{active}(\mathsf{b})$ then the second rule is not applicable. Thus, we have to add the rule $\mathsf{active}(x) \to x$. To summarize, we obtain the following transformation.

Definition 1. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}^1_\mu$ over the signature $\mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}\}$ consists of the following rewrite rules:*

$$\begin{aligned} \mathsf{active}(l) &\to \mathsf{mark}(r) && \text{for all } l \to r \in \mathcal{R} \\ \mathsf{mark}(f(x_1, \ldots, x_n)) &\to \mathsf{active}(f([x_1]_f, \ldots, [x_n]_f)) && \text{for all } f \in \mathcal{F} \\ \mathsf{active}(x) &\to x \end{aligned}$$

Here $[x_i]_f = \mathsf{mark}(x_i)$ if $i \in \mu(f)$ and $[x_i]_f = x_i$ otherwise. The subset of $\mathcal{R}^1_\mu$ consisting of all rules of the form

$$\mathsf{mark}(f(x_1, \ldots, x_n)) \to \mathsf{active}(f([x_1]_f, \ldots, [x_n]_f))$$

will be denoted by $\mathcal{M}$.

Soundness of our transformation is an easy consequence of the following lemma which shows how context-sensitive reduction steps are simulated in the transformed system.

Lemma 1. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$ and let $s, t \in \mathcal{T}(\mathcal{F})$. If $s \rightarrow_{\mathcal{R},\mu} t$ then $\mathsf{mark}(s){\downarrow_{\mathcal{M}}} \rightarrow^{+}_{\mathcal{R}^1_\mu} \mathsf{mark}(t){\downarrow_{\mathcal{M}}}$.*

Proof. First note that $\mathcal{M}$ is confluent and terminating, so $u{\downarrow_{\mathcal{M}}}$ exists for every term u. There exist a rewrite rule $l \to r \in \mathcal{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. We prove the lemma by induction on π. If $\pi = \varepsilon$ then $s = l\sigma$ and $t = r\sigma$. An easy induction on the structure of s reveals that $\mathsf{mark}(s){\downarrow_{\mathcal{M}}} \rightarrow^{*}_{\mathcal{R}^1_\mu} \mathsf{active}(s)$ (one just has to eliminate all inner occurrences of active in $\mathsf{mark}(s){\downarrow_{\mathcal{M}}}$). Since $\mathsf{active}(s) \to \mathsf{mark}(t)$ is an instance of a rule in $\mathcal{R}^1_\mu$ we obtain

$$\mathsf{mark}(s){\downarrow_{\mathcal{M}}} \rightarrow^{*}_{\mathcal{R}^1_\mu} \mathsf{active}(s) \rightarrow_{\mathcal{R}^1_\mu} \mathsf{mark}(t) \rightarrow^{+}_{\mathcal{R}^1_\mu} \mathsf{mark}(t){\downarrow_{\mathcal{M}}}.$$

If $\pi = i{\cdot}\pi'$ then we have $s = f(s_1, \dots, s_i, \dots, s_n)$ and $t = f(s_1, \dots, t_i, \dots, s_n)$ with $s_i \rightarrow_{\mathcal{R},\mu} t_i$. Note that $i \in \mu(f)$ due to the definition of context-sensitive rewriting. For $1 \leqslant j \leqslant n$ define $s'_j = \mathsf{mark}(s_j){\downarrow_{\mathcal{M}}}$ if $j \in \mu(f)$ and $s'_j = s_j$ if $j \notin \mu(f)$. The induction hypothesis yields $s'_i \rightarrow^{+}_{\mathcal{R}^1_\mu} \mathsf{mark}(t_i){\downarrow_{\mathcal{M}}}$. Since

$$\mathsf{mark}(s){\downarrow_{\mathcal{M}}} = \mathsf{active}(f(s'_1, \dots, s'_i, \dots, s'_n))$$

and

$$\mathsf{mark}(t){\downarrow_{\mathcal{M}}} = \mathsf{active}(f(s'_1, \dots, \mathsf{mark}(t_i){\downarrow_{\mathcal{M}}}, \dots, s'_n)),$$

the result follows. □

Theorem 1. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. If $\mathcal{R}^1_\mu$ is terminating then $(\mathcal{R}, \mu)$ is terminating.*

Proof. If $(\mathcal{R}, \mu)$ is not terminating then there exists an infinite reduction of ground terms. Any such sequence is transformed by the previous lemma into an infinite reduction in $\mathcal{R}^1_\mu$. □

The converse of the above theorem does not hold, i.e., the transformation is incomplete.

Example 1. As an example of a terminating CSRS that is transformed into a non-terminating TRS by our transformation, consider the following variant $\mathcal{R}$ of a well-known example from Toyama [15]:

$$\mathsf{f}(\mathsf{b}, \mathsf{c}, x) \to \mathsf{f}(x, x, x) \qquad \mathsf{d} \to \mathsf{b} \qquad \mathsf{d} \to \mathsf{c}$$

If we define $\mu(f) = \{3\}$ then the resulting CSRS is terminating because the usual cyclic reduction of $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ to $\mathsf{f}(\mathsf{d}, \mathsf{d}, \mathsf{d})$ and further to $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ cannot be done any more, as one would have to reduce the first and second argument of f. However, the transformed TRS $\mathcal{R}_\mu^1$

$$\begin{array}{llll}
\mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},x)) & \to \mathsf{mark}(\mathsf{f}(x,x,x)) & \mathsf{mark}(\mathsf{f}(x,y,z)) & \to \mathsf{active}(\mathsf{f}(x,y,\mathsf{mark}(z))) \\
\mathsf{active}(\mathsf{d}) & \to \mathsf{mark}(\mathsf{b}) & \mathsf{mark}(\mathsf{b}) & \to \mathsf{active}(\mathsf{b}) \\
\mathsf{active}(\mathsf{d}) & \to \mathsf{mark}(\mathsf{c}) & \mathsf{mark}(\mathsf{c}) & \to \mathsf{active}(\mathsf{c}) \\
\mathsf{active}(x) & \to x & \mathsf{mark}(\mathsf{d}) & \to \mathsf{active}(\mathsf{d})
\end{array}$$

is not terminating:

$$\begin{array}{rll}
\mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d})) & \to & \mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{mark}(\mathsf{d}))) \\
& \to & \mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{active}(\mathsf{d}))) \\
& \to & \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}))) \\
& \to^+ & \mathsf{mark}(\mathsf{f}(\mathsf{mark}(\mathsf{b}),\mathsf{mark}(\mathsf{c}),\mathsf{d})) \\
& \to^+ & \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{b}),\mathsf{active}(\mathsf{c}),\mathsf{d})) \\
& \to^+ & \mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d}))
\end{array}$$

Note that $\mathcal{R}_\mu^{\mathrm{L}}$:

$$\mathsf{f}_\mu(x) \to \mathsf{f}_\mu(x) \qquad \mathsf{d}_\mu \to \mathsf{b}_\mu \qquad \mathsf{d}_\mu \to \mathsf{c}_\mu$$

and $\mathcal{R}_\mu^{\mathrm{Z}}$:

$$\begin{array}{rlrl}
\mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},x) & \to \mathsf{f}(x,x,x) & \mathsf{a}(\underline{\mathsf{b}}) & \to \mathsf{b} \\
\mathsf{d} & \to \mathsf{b} & \mathsf{a}(\underline{\mathsf{c}}) & \to \mathsf{c} \\
\mathsf{d} & \to \mathsf{c} & \mathsf{b} & \to \underline{\mathsf{b}} \\
\mathsf{a}(x) & \to x & \mathsf{c} & \to \underline{\mathsf{c}}
\end{array}$$

also fail to be terminating ($\mathcal{R}_\mu^{\mathrm{Z}}$ admits the cycle $\mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},\mathsf{d}) \to \mathsf{f}(\mathsf{d},\mathsf{d},\mathsf{d}) \to^+ \mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d}) \to^+ \mathsf{f}(\underline{\mathsf{b}},\underline{\mathsf{c}},\mathsf{d})$).

Nevertheless, compared to the transformations of Lucas and Zantema, our easy transformation appears to be very powerful. There are numerous CSRSs where our transformation succeeds and which cannot be handled by the other two transformations.

Example 2. As a simple example, consider the terminating CSRS $\mathcal{R}$

$$\begin{array}{rl}
\mathsf{g}(x) & \to \mathsf{h}(x) \\
\mathsf{c} & \to \mathsf{d} \\
\mathsf{h}(\mathsf{d}) & \to \mathsf{g}(\mathsf{c})
\end{array}$$

with $\mu(\mathsf{g}) = \mu(\mathsf{h}) = \varnothing$ from [17]. The TRSs $\mathcal{R}_\mu^{\mathrm{L}}$:

$$\mathsf{g}_\mu \to \mathsf{h}_\mu \qquad \mathsf{c}_\mu \to \mathsf{d}_\mu \qquad \mathsf{h}_\mu \to \mathsf{g}_\mu$$

and $\mathcal{R}_\mu^{\mathrm{Z}}$:

$$\begin{array}{rlrl}
\mathsf{g}(x) & \to \mathsf{h}(\mathsf{a}(x)) & \mathsf{a}(\underline{\mathsf{c}}) & \to \mathsf{c} \\
\mathsf{c} & \to \mathsf{d} & \mathsf{a}(\underline{\mathsf{d}}) & \to \mathsf{d} \\
\mathsf{h}(\underline{\mathsf{d}}) & \to \mathsf{g}(\underline{\mathsf{c}}) & \mathsf{c} & \to \underline{\mathsf{c}} \\
\mathsf{a}(x) & \to x & \mathsf{d} & \to \underline{\mathsf{d}}
\end{array}$$

are non-terminating ($\mathcal{R}_\mu^{\mathrm{Z}}$ admits the cycle $\mathsf{g}(\underline{\mathsf{c}}) \to \mathsf{h}(\mathsf{a}(\underline{\mathsf{c}})) \to \mathsf{h}(\mathsf{c}) \to \mathsf{h}(\mathsf{d}) \to \mathsf{h}(\underline{\mathsf{d}}) \to \mathsf{g}(\underline{\mathsf{c}})$). In contrast, our simple transformation generates the TRS

$$\begin{array}{llll} \mathsf{active}(\mathsf{g}(x)) \to \mathsf{mark}(\mathsf{h}(x)) & & \mathsf{mark}(\mathsf{g}(x)) \to \mathsf{active}(\mathsf{g}(x)) \\ \mathsf{active}(\mathsf{c}) \to \mathsf{mark}(\mathsf{d}) & & \mathsf{mark}(\mathsf{h}(x)) \to \mathsf{active}(\mathsf{h}(x)) \\ \mathsf{active}(\mathsf{h}(\mathsf{d})) \to \mathsf{mark}(\mathsf{g}(\mathsf{c})) & & \mathsf{mark}(\mathsf{c}) \to \mathsf{active}(\mathsf{c}) \\ \mathsf{active}(x) \to x & & \mathsf{mark}(\mathsf{d}) \to \mathsf{active}(\mathsf{d}) \end{array}$$

which is terminating.[1]

Moreover, while the techniques of Lucas and Zantema fail for the Sieve of Eratosthenes example from the introduction, our transformation generates a terminating TRS. In fact, we do not know of any example where the method of Lucas or Zantema works but our method fails. (In particular, our transformation succeeds for all terminating CSRSs presented in [17].) This strongly suggests that our proposal is more powerful than the previous two approaches. For the transformation of Lucas this can indeed be proved.

Theorem 2. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. If $\mathcal{R}_\mu^{\mathrm{L}}$ is terminating then $\mathcal{R}_\mu^1$ is terminating.*

Proof. We prove termination of $\mathcal{R}_\mu^1$ using the *dependency pair* approach of Arts and Giesl [1–3]. The dependency pairs of $\mathcal{R}_\mu^1$ are

$$\begin{array}{lll} \langle \mathsf{ACTIVE}(l), \mathsf{MARK}(r)\rangle & \text{for all } l \to r \text{ in } \mathcal{R} & \text{(i)} \\ \langle \mathsf{MARK}(f(x_1,\dots,x_n)), \mathsf{ACTIVE}(f([x_1]_f,\dots,[x_n]_f))\rangle & \text{for all } f \in \mathcal{F} & \text{(ii)} \\ \langle \mathsf{MARK}(f(x_1,\dots,x_n)), \mathsf{MARK}(x_i)\rangle & \text{for } f \in \mathcal{F},\, i \in \mu(f) & \text{(iii)} \end{array}$$

To prove termination of $\mathcal{R}_\mu^1$ we have to find a weakly monotonic quasi-order $\succsim$ and a well-founded order $\succ$ which is compatible with $\succsim$ (i.e., $\succ \circ \succsim \subseteq \succ$) such that both $\succ$ and $\succsim$ are closed under substitution. Then it is sufficient if the following constraints are satisfied. Dependency pairs of kind (i) and (iii) should be strictly decreasing and for dependency pairs of kind (ii) it is enough if they are weakly decreasing. Moreover, all rules of $\mathcal{R}_\mu^1$ should be weakly decreasing. Thus, we only have to demand

$$\begin{array}{rll} \mathsf{ACTIVE}(l) \succ \mathsf{MARK}(r) & & \text{for all } l \to r \text{ in } \mathcal{R} \\ \mathsf{MARK}(f(x_1,\dots,x_n)) \succsim \mathsf{ACTIVE}(f([x_1]_f,\dots,[x_n]_f)) & & \text{for all } f \in \mathcal{F} \\ \mathsf{MARK}(f(x_1,\dots,x_n)) \succ \mathsf{MARK}(x_i) & & \text{for all } f \in \mathcal{F},\, i \in \mu(f) \\ \mathsf{active}(l) \succsim \mathsf{mark}(r) & & \text{for all } l \to r \text{ in } \mathcal{R} \\ \mathsf{mark}(f(x_1,\dots,x_n)) \succsim \mathsf{active}(f([x_1]_f,\dots,[x_n]_f)) & & \text{for all } f \in \mathcal{F} \\ \mathsf{active}(x) \succsim x & & \end{array}$$

[1] This can be proved using the *dependency pair* approach ([3]): Since the pair $\langle \mathsf{ACTIVE}(\mathsf{h}(\mathsf{d})), \mathsf{MARK}(\mathsf{g}(\mathsf{c}))\rangle$ can occur at most once in any *chain* of dependency pairs, it follows that there are no infinite chains and hence the TRS is terminating.

Let $\mathcal{A}$ be the (confluent and terminating) TRS consisting of the rewrite rules

$$\begin{aligned} \mathsf{ACTIVE}(x) &\to x \\ \mathsf{MARK}(x) &\to x \\ \mathsf{active}(x) &\to x \\ \mathsf{mark}(x) &\to x \\ f(x_1,\ldots,x_n) &\to f_\mu(x_{i_1},\ldots,x_{i_k}) \end{aligned}$$

for all $f \in \mathcal{F}$ where $\mu(f) = \{i_1,\ldots,i_k\}$ with $i_1 < \cdots < i_k$. Define $s \succ t$ if and only if $s{\downarrow_\mathcal{A}} \, (\to_{\mathcal{R}^{\mathrm{L}}_\mu} \cup \rhd)^+ \, t{\downarrow_\mathcal{A}}$. Here $\rhd$ denotes the proper subterm relation. Moreover, let $s \succsim t$ hold if and only if $s{\downarrow_\mathcal{A}} \to^*_{\mathcal{R}^{\mathrm{L}}_\mu} t{\downarrow_\mathcal{A}}$. One easily verifies that $\succ$ and $\succsim$ satisfy the above demands ($\succ$ is well founded by the termination of $\mathcal{R}^{\mathrm{L}}_\mu$). Hence, due to the soundness of the dependency pair approach, the termination of $\mathcal{R}^1_\mu$ is established. $\square$

This theorem can also be proved using the *self-labelling* technique of [12].

4 A Sound and Complete Transformation

In this section we present a transformation of context-sensitive rewrite systems which is not only sound but also complete with respect to termination. To appreciate the non-triviality of this result, the reader may want to try to construct a sound and complete transformation (together with a proof of completeness) before reading any further.

Let us first investigate why the transformation of Sect. 3 lacks completeness. Consider again the CSRS $(\mathcal{R},\mu)$ of Example 1. The reason for the non-termination of $\mathcal{R}^1_\mu$ is that terms may have occurrences of active at forbidden positions, even if we start with a "proper" term (like $\mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d}))$). The "forbidden" occurrences of active in the first two arguments of f (in the term $\mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d})))$) lead to contractions which are impossible in the underlying CSRS. Thus, the key to achieving a complete transformation is to control the number of occurrences of active. We do this in a rather drastic manner: we will work with a single occurrence of active. Of course, we cannot forbid the existence of terms with multiple occurrences of active but we can make sure that no new active symbols are introduced during the contraction of an active redex.

Working with a single active occurrence entails that we have to shift it in a non-deterministic fashion downwards to any active position. This is achieved by the rules

$$\mathsf{active}(f(x_1,\ldots,x_i,\ldots,x_n)) \to f'(x_1,\ldots,\mathsf{active}(x_i),\ldots,x_n)$$

for every $i \in \mu(f)$. When shifting the active symbol to an argument of f, the original function symbol f is replaced by a new function symbol f'. This is to ensure that no reductions can take place above the current position of active. By this shifting of the symbol active, our TRS implements an algorithm to search for redexes subject to the constraints of the replacement map μ. Once we have shifted active to the position of the desired redex, we can apply one of the rules

$$\mathsf{active}(l) \to \mathsf{mark}(r)$$

as in the previous transformation. The function symbol mark is used to mark the contractum of the selected redex. In order to continue the reduction it has to be replaced by active again. Since the next reduction step may of course take place at a position above the previously contracted redex, we first have to shift mark upwards through the term, i.e., we use rules of the form

$$f'(x_1, \ldots, \mathsf{mark}(x_i), \ldots, x_n) \to \mathsf{mark}(f(x_1, \ldots, x_i, \ldots, x_n))$$

for every $i \in \mu(f)$. We want to replace mark by active if there are no f' symbols left above it. Since the absence of f' symbols cannot be determined, we introduce a new unary function symbol top to mark the position below which reductions may take place. Thus, the reduction of a term s with respect to a CSRS is modelled by the reduction of the term $\mathsf{top}(\mathsf{active}(s))$ in the transformed TRS. If $\mathsf{top}(\mathsf{active}(s))$ is reduced to a term $\mathsf{top}(\mathsf{mark}(t))$, we are ready to replace mark by active. This suggests adding the rule

$$\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{active}(x)).$$

However, as illustrated with the counterexample in Sect. 3, we have to avoid making infinite reductions with terms which contain inner occurrences of new symbols like active and mark. For that reason we want to make sure that this rule is only applicable to terms that do not contain any other occurrences of the new function symbols. Thus, before reducing $\mathsf{top}(\mathsf{mark}(t))$ to $\mathsf{top}(\mathsf{active}(t))$ we check whether the term t is *proper*, i.e., whether it contains only function symbols from the original signature $\mathcal{F}$. This is easily achieved by new unary function symbols proper and ok. For any ground term $t \in \mathcal{T}(\mathcal{F})$, $\mathsf{proper}(t)$ reduces to $\mathsf{ok}(t)$, but if t contains one of the newly introduced function symbols then the reduction of $\mathsf{proper}(t)$ is blocked. This is done by the rules

$$\mathsf{proper}(c) \to \mathsf{ok}(c)$$

for every constant $c \in \mathcal{F}$ and

$$\begin{aligned}\mathsf{proper}(f(x_1, \ldots, x_n)) &\to f(\mathsf{proper}(x_1), \ldots, \mathsf{proper}(x_n))\\ f(\mathsf{ok}(x_1), \ldots, \mathsf{ok}(x_n)) &\to \mathsf{ok}(f(x_1, \ldots, x_n))\end{aligned}$$

for every function symbol $f \in \mathcal{F}$ of arity $n > 0$. Now, instead of the rule $\mathsf{top}(\mathsf{mark}(x)) \to \mathsf{top}(\mathsf{active}(x))$ we adopt the rules

$$\begin{aligned}\mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{proper}(x))\\ \mathsf{top}(\mathsf{ok}(x)) &\to \mathsf{top}(\mathsf{active}(x)).\end{aligned}$$

This concludes our informal explanation of the new transformation, whose formal definition is summarized below.

Definition 2. *Let $(\mathcal{R},\mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}^2_\mu$ over the signature $\mathcal{F}' = \mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}, \mathsf{top}, \mathsf{proper}, \mathsf{ok}\} \cup \{f' \mid f \in \mathcal{F}$ is not a constant$\}$ consists of the following rewrite rules (for all $l \to r \in \mathcal{R}$, $f \in \mathcal{F}$ of arity $n > 0$, $i \in \mu(f)$, and constants $c \in \mathcal{F}$):*

$$
\begin{aligned}
\mathsf{active}(l) &\to \mathsf{mark}(r)\\
\mathsf{active}(f(x_1,\ldots,x_i,\ldots,x_n)) &\to f'(x_1,\ldots,\mathsf{active}(x_i),\ldots,x_n)\\
f'(x_1,\ldots,\mathsf{mark}(x_i),\ldots,x_n) &\to \mathsf{mark}(f(x_1,\ldots,x_i,\ldots,x_n))\\
\mathsf{proper}(c) &\to \mathsf{ok}(c)\\
\mathsf{proper}(f(x_1,\ldots,x_n)) &\to f(\mathsf{proper}(x_1),\ldots,\mathsf{proper}(x_n))\\
f(\mathsf{ok}(x_1),\ldots,\mathsf{ok}(x_n)) &\to \mathsf{ok}(f(x_1,\ldots,x_n))\\
\mathsf{top}(\mathsf{mark}(x)) &\to \mathsf{top}(\mathsf{proper}(x))\\
\mathsf{top}(\mathsf{ok}(x)) &\to \mathsf{top}(\mathsf{active}(x))
\end{aligned}
$$

In the remainder of this section we show that our second transformation is both sound and complete. We start with a preliminary lemma, which states that proper has indeed the desired effect.

Lemma 2. *Let $(\mathcal{R},\mu)$ be a CSRS over a signature $\mathcal{F}$. Let $s,t \in \mathcal{T}(\mathcal{F}')$. We have $\mathsf{proper}(s) \to^+_{\mathcal{R}^2_\mu} \mathsf{ok}(t)$ if and only if $s = t$ and $s \in \mathcal{T}(\mathcal{F})$.*

Proof. The "if" direction is an easy induction proof on the structure of s. The "only if" direction can be proved by induction on the number of symbols in s.

If the root of s is a function symbol $g \in \mathcal{F}' \setminus (\mathcal{F} \cup \{\mathsf{proper}\})$ then $\mathsf{proper}(s)$ cannot be rewritten at the root. Thus, any one-step reduction of $\mathsf{proper}(s)$ would yield a term of the form $\mathsf{proper}(s')$ where $s \to_{\mathcal{R}^2_\mu} s'$. If $g \in \{\mathsf{active}, \mathsf{mark}\} \cup \{f' \mid f \in \mathcal{F}$ is not a constant$\}$ then the root symbol of s' must also be from that set. Similarly, if g is ok or top, then the root symbol of s' is g as well. This implies that no reduct of $\mathsf{proper}(s)$ can be reduced at the root position either. Hence $\mathsf{proper}(s) \to^+_{\mathcal{R}^2_\mu} \mathsf{ok}(t)$ cannot hold and the claim holds vacuously.

In the remaining case the root symbol of s is from $\mathcal{F} \cup \{\mathsf{proper}\}$. Thus, s has the form $\mathsf{proper}^m(u)$ for some $m \geqslant 0$ where the root of u is different from proper. In order to reduce $\mathsf{proper}(s)$ at the root, we first have to reduce $s = \mathsf{proper}^m(u)$ to a term with a root symbol from $\mathcal{F}$. Similar to the observations above, the root symbol of u cannot be from $\mathcal{F}' \setminus \mathcal{F}$. If u is a constant from $\mathcal{F}$ then the only applicable rule is $\mathsf{proper}(u) \to \mathsf{ok}(u)$. Thus, $\mathsf{proper}(s) = \mathsf{proper}^{m+1}(u)$ is reduced to the normal form $\mathsf{proper}^m(\mathsf{ok}(u))$. So in this case $\mathsf{proper}(s)$ can only rewrite to a term of the form $\mathsf{ok}(t)$ if $m = 0$ and thus the claim of the lemma holds trivially.

Otherwise, $u = f(u_1,\ldots,u_n)$ with $f \in \mathcal{F}$ of arity $n > 0$. The reduction from $\mathsf{proper}(s)$ to $\mathsf{ok}(t)$ must start as follows:

$$
\begin{aligned}
\mathsf{proper}(s) \;&=\; \mathsf{proper}(\mathsf{proper}^m(f(u_1,\ldots,u_n)))\\
&\to^*_{\mathcal{R}^2_\mu} \mathsf{proper}(\mathsf{proper}^m(f(u'_1,\ldots,u'_n)))\\
&\to_{\mathcal{R}^2_\mu} \mathsf{proper}(\mathsf{proper}^{m-1}(f(\mathsf{proper}(u'_1),\ldots,\mathsf{proper}(u'_n))))\\
&\to^*_{\mathcal{R}^2_\mu} \cdots
\end{aligned}
$$

$$\begin{aligned}
&\to_{\mathcal{R}^2_\mu} \mathsf{proper}(f(u''_1, \ldots, u''_n)) \\
&\to^*_{\mathcal{R}^2_\mu} f(\mathsf{proper}(u''_1), \ldots, \mathsf{proper}(u''_n))
\end{aligned}$$

where $\mathsf{proper}^m(u_i) \to^*_{\mathcal{R}^2_\mu} u''_i$ for all $1 \leqslant i \leqslant n$. (Note that the root symbol f of u must not be rewritten to ok, for otherwise no reduction step at the root can take place.) To reduce $f(\mathsf{proper}(u''_1), \ldots, \mathsf{proper}(u''_n))$ to a term of the form $\mathsf{ok}(t)$, every argument $\mathsf{proper}(u''_i)$ must be reduced to a term of the form $\mathsf{ok}(t_i)$ and then $f(\mathsf{ok}(t_1), \ldots, \mathsf{ok}(t_n))$ can be reduced to $\mathsf{ok}(f(t_1, \ldots, t_n))$. But if $\mathsf{proper}(u''_i) \to^*_{\mathcal{R}^2_\mu} \mathsf{ok}(t_i)$ then we also have $\mathsf{proper}(\mathsf{proper}^m(u_i)) \to^*_{\mathcal{R}^2_\mu} \mathsf{ok}(t_i)$. The induction hypothesis yields $\mathsf{proper}^m(u_i) = t_i$ and $\mathsf{proper}^m(u_i) \in \mathcal{T}(\mathcal{F})$ for all $1 \leqslant i \leqslant n$. So in this case we have $m = 0$ as well, i.e., s cannot contain any occurrence of proper. Consequently, $\mathsf{ok}(f(t_1, \ldots, t_n))$ is in normal form and hence $s = u = f(u_1, \ldots, u_n) = f(t_1, \ldots, t_n) = t \in \mathcal{T}(\mathcal{F})$. □

The next lemma shows how context-sensitive reduction steps are simulated by the second transformation. The "if" part is used in the completeness proof.

Lemma 3. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$ and let $s \in \mathcal{T}(\mathcal{F})$. We have $s \to_{\mathcal{R},\mu} t$ if and only if $\mathsf{active}(s) \to^+_{\mathcal{R}^2_\mu} \mathsf{mark}(t)$.*

Proof. The "only if" direction is easily proved by induction on the depth of the position of the redex contracted in $s \to_{\mathcal{R},\mu} t$. We prove here the "if" direction by induction on s. There are two possibilities for the rewrite rule of $\mathcal{R}^2_\mu$ that is applied in the first step of the reduction from $\mathsf{active}(s)$ to $\mathsf{mark}(t)$. If a rule of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ is used, then $s = l\sigma$ for some substitution σ. Since $r\sigma$ contains only symbols from $\mathcal{F}$, $\mathsf{mark}(r\sigma)$ is in normal form and thus $t = r\sigma$. Clearly $s \to_{\mathcal{R},\mu} t$.

Otherwise, s must have the form $f(s_1, \ldots, s_i, \ldots, s_n)$ and in the first reduction step $\mathsf{active}(s)$ is reduced to $f'(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n)$ for some $i \in \mu(f)$. Note that all reductions of the latter term to a term of the form $\mathsf{mark}(t)$ have the form

$$\begin{aligned}
f'(s_1, \ldots, \mathsf{active}(s_i), \ldots, s_n) &\to^+_{\mathcal{R}^2_\mu} f'(s_1, \ldots, \mathsf{mark}(t_i), \ldots, s_n) \\
&\to_{\mathcal{R}^2_\mu} \mathsf{mark}(f(s_1, \ldots, t_i, \ldots, s_n)).
\end{aligned}$$

Hence $t = f(s_1, \ldots, t_i, \ldots, s_n)$. The induction hypothesis yields $s_i \to_{\mathcal{R},\mu} t_i$ and as $i \in \mu(f)$ we also have $s \to_{\mathcal{R},\mu} t$. □

Soundness of our second transformation is now easily shown.

Theorem 3. *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. If $\mathcal{R}^2_\mu$ is terminating then $(\mathcal{R}, \mu)$ is terminating.*

Proof. If $(\mathcal{R}, \mu)$ is not terminating then there exists an infinite reduction of ground terms in $\mathcal{T}(\mathcal{F})$. Note that $s \to_{\mathcal{R},\mu} t$ implies $\mathsf{active}(s) \to^+_{\mathcal{R}^2_\mu} \mathsf{mark}(t)$ by Lemma 3. Hence it also implies

$$\mathsf{top}(\mathsf{active}(s)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{mark}(t)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{proper}(t)).$$

Moreover, by Lemma 2 we have $\mathsf{proper}(t) \to^+_{\mathcal{R}^2_\mu} \mathsf{ok}(t)$ and thus

$$\mathsf{top}(\mathsf{proper}(t)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{ok}(t)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{active}(t)).$$

Concatenating these two reductions shows that $\mathsf{top}(\mathsf{active}(s)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{active}(t))$ whenever $s \to_{\mathcal{R},\mu} t$. Hence any infinite reduction of ground terms in $(\mathcal{R},\mu)$ is transformed into an infinite reduction in $\mathcal{R}^2_\mu$. ☐

To prove that the converse of Theorem 3 holds as well, we define $\mathcal{S}^2_\mu$ as the TRS $\mathcal{R}^2_\mu$ without the two rewrite rules for top. The following lemma states that we do not have to worry about $\mathcal{S}^2_\mu$.

Lemma 4. *The TRS $\mathcal{S}^2_\mu$ is terminating for any CSRS $(\mathcal{R},\mu)$.*

Proof. Let $\mathcal{F}$ be the signature of $(\mathcal{R},\mu)$. The rewrite rules of $\mathcal{S}^2_\mu$ are oriented from left to right by $\succ_{\mathrm{rpo}}$, the recursive path order [5] induced by the following precedence $\succ$ on $\mathcal{F}'$:

$$\mathsf{active} \succ f' \succ \mathsf{mark} \succ \mathsf{proper} \succ f \succ c \succ \mathsf{ok}$$

for every non-constant $f \in \mathcal{F}$ and every constant $c \in \mathcal{F}$. Since $\succ$ is well-founded, it follows that $\mathcal{S}^2_\mu$ is terminating. ☐

Now we are ready to present the main theorem of the paper.

Theorem 4. *Let $(\mathcal{R},\mu)$ be a CSRS over a signature $\mathcal{F}$. If $(\mathcal{R},\mu)$ is terminating then $\mathcal{R}^2_\mu$ is terminating.*

Proof. First note that the precedence used in the proof of Lemma 4 cannot be extended to deal with the whole of $\mathcal{R}^2_\mu$ as the second rewrite rule for top requires $\mathsf{ok} \succ \mathsf{active}$. Since $\mathcal{R}^2_\mu$ lacks collapsing rules, it is sufficient to prove termination of any *typed* version of $\mathcal{R}^2_\mu$, cf. [16, 13]. Thus we may assume that the function symbols of $\mathcal{R}^2_\mu$ come from a many-sorted signature, where the only restriction is that the left and right-hand side of any rewrite rule are well-typed and of the same type. We use two sorts α and β, with top of type $\alpha \to \beta$ and all other symbols of type $\alpha \times \ldots \times \alpha \to \alpha$. So if $\mathcal{R}^2_\mu$ allows an infinite reduction then there exists an infinite reduction of well-typed terms. Since both types contain a ground term, we may assume for a proof by contradiction that there exists an infinite reduction starting from a well-typed *ground* term t. Terms of type α are terminating by Lemma 4 since they cannot contain the symbol top and thus the only applicable rules stem from $\mathcal{S}^2_\mu$. So t is a ground term of type β, which implies that $t = \mathsf{top}(t')$ with t' of type α. Since t' is terminating, the infinite reduction starting from t must contain a root reduction step. So t' reduces to $\mathsf{mark}(t_1)$ or $\mathsf{ok}(t_1)$ for some term t_1 (of type α). We consider the former possibility, the latter possibility is treated in a very similar way. The infinite reduction starts with

$$t \to^*_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{mark}(t_1)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{proper}(t_1)).$$

Since $\mathsf{proper}(t_1)$ is of type α and thus terminating, after some further reduction steps another step takes place at the root. This is only possible if $\mathsf{proper}(t_1)$ reduces to $\mathsf{ok}(t_2)$ for some term t_2. According to Lemma 2 we must have $t_1 = t_2 \in \mathcal{T}(\mathcal{F})$. Hence the presupposed infinite reduction continues as follows:

$$\mathsf{top}(\mathsf{proper}(t_1)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{ok}(t_1)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{active}(t_1)).$$

Repeating this kind of reasoning reveals that the infinite reduction must be of the following form, where all root reduction steps between $\mathsf{top}(\mathsf{proper}(t_1))$ and $\mathsf{top}(\mathsf{mark}(t_3))$ are made explicit:

$$\begin{aligned} t &\to^*_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{proper}(t_1)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{ok}(t_1)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{active}(t_1)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{mark}(t_2)) \\ &\to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{proper}(t_2)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{ok}(t_2)) \to_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{active}(t_2)) \to^+_{\mathcal{R}^2_\mu} \mathsf{top}(\mathsf{mark}(t_3)) \\ &\to_{\mathcal{R}^2_\mu} \cdots \end{aligned}$$

Hence $\mathsf{active}(t_i) \to^+_{\mathcal{R}^2_\mu} \mathsf{mark}(t_{i+1})$ and $t_i \in \mathcal{T}(\mathcal{F})$ for all $i \geqslant 1$. We obtain

$$t_1 \to_{\mathcal{R},\mu} t_2 \to_{\mathcal{R},\mu} t_3 \to_{\mathcal{R},\mu} \cdots$$

from Lemma 3, contradicting the termination of $(\mathcal{R}, \mu)$. □

5 Conclusion and Further Work

In this paper we presented two new transformations from CSRSs to TRSs whose purpose is to reduce the problem of proving termination of CSRSs to the problem of proving termination of TRSs. The advantage of such an approach is that all termination techniques for ordinary term rewriting (including future developments) become available for context-sensitive rewriting as well. So in particular, these techniques can now also be used to analyze the termination behaviour of lazy functional programs which may be modelled by CSRSs. Our first transformation is simple, sound, and appears to be more powerful than previously suggested transformations. Our second transformation is not only sound but also complete, so it transforms every terminating CSRS into a terminating TRS.

Our transformations also form a basis for *automated* termination proofs of CSRSs. Of course, a direct termination proof of $\mathcal{R}^2_\mu$ cannot be obtained by a path order amenable to automation and even a powerful method like the dependency pair approach often will not succeed in finding a fully automated termination proof. To a lesser extent this is already true for our first transformation. However, our transformations are suitable for changes in their presentation which do not result in any significant change in their behaviour, but which ease the termination proofs of the resulting TRSs considerably.

For instance, for the first transformation an obvious idea is to normalize the right-hand sides of the $\mathsf{active}(l) \to \mathsf{mark}(r)$ rules with respect to the subsystem $\mathcal{M}$. Another natural idea is to replace the single symbol active by fresh symbols f_{active} for every $f \in \mathcal{F}$. This amounts to replacing every occurrence of the pattern

$\mathsf{active}(f(\cdots))$ in the rewrite rules by $f_{\mathsf{active}}(\cdots)$ as well as expanding the rule $\mathsf{active}(x) \to x$ into all rules of the form $f_{\mathsf{active}}(x_1, \ldots, x_n) \to f(x_1, \ldots, x_n)$. If we apply both ideas to the TRS $\mathcal{R}^1_\mu$ of Example 2 we obtain the TRS

$$
\begin{array}{lll}
\mathsf{g}_{\mathsf{active}}(x) \to \mathsf{h}_{\mathsf{active}}(x) & \mathsf{mark}(\mathsf{g}(x)) \to \mathsf{g}_{\mathsf{active}}(x) & \mathsf{g}_{\mathsf{active}}(x) \to \mathsf{g}(x) \\
\mathsf{c}_{\mathsf{active}} \quad\;\; \to \mathsf{d}_{\mathsf{active}} & \mathsf{mark}(\mathsf{h}(x)) \to \mathsf{h}_{\mathsf{active}}(x) & \mathsf{h}_{\mathsf{active}}(x) \to \mathsf{h}(x) \\
\mathsf{h}_{\mathsf{active}}(\mathsf{d}) \to \mathsf{g}_{\mathsf{active}}(\mathsf{c}) & \mathsf{mark}(\mathsf{c}) \quad\;\; \to \mathsf{c}_{\mathsf{active}} & \mathsf{c}_{\mathsf{active}} \quad\;\; \to \mathsf{c} \\
 & \mathsf{mark}(\mathsf{d}) \quad\;\; \to \mathsf{d}_{\mathsf{active}} & \mathsf{d}_{\mathsf{active}} \quad\;\; \to \mathsf{d}
\end{array}
$$

which is compatible with $\succ_{\mathrm{rpo}}$ for the precedence $\mathsf{mark} \succ \mathsf{c}_{\mathsf{active}} \succ \mathsf{d}_{\mathsf{active}} \succ \mathsf{d} \succ \mathsf{c} \succ \mathsf{g}_{\mathsf{active}} \succ \mathsf{g} \succ \mathsf{h}_{\mathsf{active}} \succ \mathsf{h}$.

Refinements like those mentioned above should be studied further. Termination of the TRS resulting from our first (incomplete) transformation is sometimes easier to prove than termination of the TRS resulting from our second (complete) one. Thus, we conclude by stating that while our second transformation is superior to all previous incomplete ones, at present our incomplete transformation of Sect. 3 as well as the ones of Lucas [10] and Zantema [17] may still be useful for the purpose of automation. In addition, the latter paper contains a complete semantic characterization of context-sensitive rewriting which can be used in a direct termination proof attempt.

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