Context-Moving Transformations for Function Verification*

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Abstract

Several induction theorem provers have been developed which support mechanized verification of functional programs. Unfortunately, a major problem is that they often fail in verifying tail recursive functions (which correspond to imperative programs). However, in practice imperative programs are used almost exclusively.

We present an automatic transformation to tackle this problem. It transforms functions which are hard to verify into functions whose correctness can be shown by the existing provers. In contrast to classical program transformations, the aim of our technique is not to increase efficiency, but to increase verifiability. Therefore, this paper introduces a novel application area for program transformations and it shows that such techniques can in fact solve some of the most urgent current challenge problems in automated verification and induction theorem proving.

1 Introduction

To guarantee the correctness of programs, a formal verification is required. However, mathematical correctness proofs are usually very expensive and time-consuming. Therefore, program verification should be *automated* as far as possible.

As *induction*[1] is the essential proof method needed for such verifications, several systems have been developed for automated induction theorem proving. These systems are successfully used for verification of *functional* programs in many areas, but a major problem for their practical application is that they are often not suitable for handling *imperative* programs. The reason is that the translation of imperative programs into the functional input language of these systems always yields *tail recursive* functions which are particularly hard to verify. Thus, developing techniques for proofs about tail recursive functions is currently one of the most important research topics in this area.

In Section 2 we present our functional programming language. We also give a brief introduction to induction theorem proving and we illustrate that the reason for the difficulties in verifying tail recursive functions is that they usually have an accumulator parameter which is initialized with some fixed value, but this value is changed in the recursive calls.

This paper introduces a new framework for mechanized verification of such functions by first transforming them into functions which are better suitable for verification and by afterwards applying the existing induction provers for their verification. To solve the verification problems with tail recursive functions, the *context* around recursive accumulator arguments has to be shifted away, such that the accumulator parameter is no longer changed in recursive calls. For

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[1]In this paper, "induction" stands for *mathematical induction*, i.e., it should not be confused with induction in the sense of machine learning.

that purpose, in Section 3 - 5 we introduce two automatic transformation techniques which transform tail recursion into non-tail recursion. Our transformations proved successful on a representative collection of tail recursive functions, cf. Appendix B. In this way, correctness of many imperative programs can be proved *automatically* without the need for inventing loop invariants or generalizations.

2 Functional Programs and their Verification

We consider a first order functional language with eager semantics and (non-parameterized and free) algebraic data types. As an example, regard the data type nat for natural numbers whose objects are built with the *constructors* 0 and $\mathsf{s} : \mathsf{nat} \to \mathsf{nat}$ (for the **s**uccessor function). Thus, the constructor ground terms represent the data objects of the respective data type. In the following, we often write "1" instead of "s(0)", etc. For every n-ary constructor c there are n *selector* functions $d_1, \ldots, d_n$ which serve as inverse functions to c (i.e., $d_i(c(x_1, \ldots, x_n)) \equiv x_i$). For example, for the unary constructor s we have the selector function p such that $\mathsf{p}(\mathsf{s}(m)) \equiv m$ (i.e., p is the **p**redecessor function).

In particular, every program F contains the type bool whose objects are built with the (nullary) constructors true and false. Moreover, there is a built-in equality function $= : \tau \times \tau \to \mathsf{bool}$ for every data type τ. To distinguish the function symbol $=$ from the equality predicate symbol, we denote the latter by "$\equiv$". The *functions* of a *functional program* F have the following form.

$$\begin{array}{l}
\textbf{function}\ \ f\,(x_1 : \tau_1, \ldots, x_n : \tau_n) : \tau \ \Leftarrow \\
\quad \textbf{if}\ b_1\ \ \textbf{then}\ r_1 \\
\qquad\qquad \vdots \\
\quad \textbf{if}\ b_m\ \textbf{then}\ r_m
\end{array}$$

Here, "**if** b_i **then** r_i" is called the i-th *case* of f with *condition* b_i and *result* r_i. For functions with just one case of the form "**if** true **then** r" we write "**function** $f\,(x_1 : \tau_1, \ldots, x_n : \tau_n) : \tau \ \Leftarrow\ r$". To ease readability, if b_m is true, then we often denote the last case by "**else** r_m". As an example, consider the following function (which calls an auxiliary algorithm $+$ for addition).

$$\begin{array}{l}
\textbf{function}\ \mathsf{times}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x \neq \mathsf{0}\ \textbf{then}\ y + \mathsf{times}(\mathsf{p}(x), y) \\
\qquad\qquad\ \textbf{else}\ \ \mathsf{0}
\end{array}$$

If a function f is called with a tuple of ground terms t^* as arguments, then t^* is evaluated first (to constructor ground terms q^*). Now the condition $b_1[x^*/q^*]$ of the first case is checked. If it evaluates to true, then $r_1[x^*/q^*]$ is evaluated. Otherwise, the condition of the second case is checked, etc. So the conditions of a functional program as above are tested from top to bottom.

Now our aim is to verify statements about the algorithms of a functional program F. In this paper we only consider universally quantified equations of the form $\forall \ldots\ s \equiv t$ and we often omit the quantifiers to ease readability. Let s, t contain the tuple of variables x^*. Then $s \equiv t$ is *inductively true* for the program F, denoted $F \models_{\mathrm{ind}} s \equiv t$, if for all those data objects q^* where evaluation of $s[x^*/q^*]$ or evaluation of $t[x^*/q^*]$ is defined, evaluation of the other term $t[x^*/q^*]$ resp. $s[x^*/q^*]$ is defined as well, and if both evaluations yield the same result. For example, let "$*$" be an abbreviation for times. Then the conjecture

$$(x * y) * z \equiv x * (y * z) \tag{1}$$

is inductively true, since $(x*y)*z$ and $x*(y*z)$ evaluate to the same result for all instantiations with data objects.

Similar notions of inductive truth are widely used in program verification and induction theorem proving. For an extension of inductive truth to more general formulas and for a model theoretic characterization (using initial algebras) see e.g. [ZKK88, Wal94, BR95, Gie99b].

To prove inductive truth automatically, several *induction theorem provers* have been developed, e.g. [BM79, KM87, ZKK88, BSH$^+$93, Wal94, BR95]. For instance, these systems can prove conjecture (1) using a *structural* induction with x as the induction variable. If we abbreviate (1) by $\varphi(x, y, z)$, then in the induction base case they would prove $\varphi(0, y, z)$ and in the step case (where $x \neq 0$), they would show that the induction hypothesis $\varphi(\mathsf{p}(x), y, z)$ implies the induction conclusion $\varphi(x, y, z)$.

However, one of the main problems for the application of these induction theorem provers in practice is that most of them can only handle functional algorithms with recursion, but they are not designed to verify imperative algorithms containing loops.

The classical techniques for the verification of imperative programs (like the so-called Hoare-calculus [Hoa69]) allow the proof of partial correctness statements of the form

$$\{\varphi_{\text{pre}}\}\ \mathcal{P}\ \{\varphi_{\text{post}}\}.$$

The semantics of this expression is that in case of termination, the program $\mathcal{P}$ transforms all program states which satisfy the precondition φ_{pre} into program states satisfying the postcondition φ_{post}. As an example, regard the following imperative program for multiplication.

$$\begin{array}{l} \textbf{procedure}\ \mathsf{multiply}\,(x, y, z : \mathsf{nat}) \Leftarrow \\ \qquad z := 0; \\ \qquad \textbf{while}\ x \neq 0\ \textbf{do} \quad x := \mathsf{p}(x); \\ \qquad\qquad\qquad\qquad\quad z := y + z \quad \textbf{od} \end{array}$$

To verify that this imperative program is equivalent to the functional program times, one has to prove the statement

$$\{x \equiv x_0 \wedge y \equiv y_0 \wedge z \equiv 0\}\ \ \textbf{while}\ x \neq 0\ \textbf{do}\ x := \mathsf{p}(x); z := y + z\ \textbf{od}\ \ \{z \equiv x_0 * y_0\}.$$

Here, x_0 and y_0 are additional variables which represent the initial values of the variables x and y. However, in the Hoare-calculus, for that purpose one needs a *loop invariant* which is a consequence of the precondition and which (together with the exit condition $x = 0$ of the loop) implies the postcondition $z \equiv x_0 * y_0$. In our example, the proof succeeds with the following loop invariant.

$$z + x * y \equiv x_0 * y_0 \tag{2}$$

The search for loop invariants is the main difficulty when verifying imperative programs. Of course, it would be desirable that programmers develop suitable loop invariants while writing their programs, but in reality this is still often not the case. Thus, for an *automation* of program verification, suitable loop invariants would have to be discovered mechanically. However, while there exist some heuristics and techniques for the choice of loop invariants [SI98], in general this task seems difficult to mechanize [Dij85].

Therefore, in the following we present an alternative approach for automated verification of imperative programs. For that purpose our aim was to use the existing powerful induction theorem provers. As the input language of these systems is restricted to functional programs, one first has to translate imperative programs into functional ones. Such a translation can easily be done automatically, cf. [McC60, Gie99a].

In this translation, every **while**-loop is transformed into a separate function. For the loop of the procedure multiply we obtain the following algorithm mult which takes the input values of x, y, and z as arguments. If the loop-condition is satisfied (i.e., if $x \neq 0$), then mult is called recursively with the new values of x, y, and z. Otherwise, mult returns the value of z. The whole imperative procedure multiply corresponds to the following functional algorithm with the same name which calls the auxiliary function mult with the initial value $z = 0$.

function multiply $(x, y : \mathsf{nat}) : \mathsf{nat} \Leftarrow$
  $\mathsf{mult}(x, y, 0)$

function mult $(x, y, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow$
  if $x \neq 0$ **then** $\mathsf{mult}(\mathsf{p}(x), y, y + z)$
  else z

Now induction provers may be used to verify conjectures about the functions multiply and mult. However, it turns out that the functional algorithms resulting from this translation have a certain characteristic form which makes them unsuitable for verification tasks. In fact, this difficulty corresponds to the problem of finding loop invariants for the original imperative program.

To verify the equivalence between multiply and times using the transformed functions multiply and mult, one now has to prove $\mathsf{multiply}(x, y) \equiv x * y$ or, in other words,

$$\mathsf{mult}(x, y, 0) \equiv x * y. \tag{3}$$

Using structural induction on x, the base formula $\mathsf{mult}(0, y, 0) \equiv 0 * y$ can easily be proved, but there is a problem with the induction step. In the case $x \neq 0$ we have to show that the induction hypothesis

$$\mathsf{mult}(\mathsf{p}(x), y, 0) \equiv \mathsf{p}(x) * y \tag{IH}$$

implies the induction conclusion $\mathsf{mult}(x, y, 0) \equiv x * y$. Using the algorithms of mult and times, the induction conclusion can be transformed into

$$\mathsf{mult}(\mathsf{p}(x), y, y) \equiv y + \mathsf{p}(x) * y. \tag{IC}$$

However, the desired proof fails, since the induction hypothesis (IH) cannot be successfully used for the proof of (IC).

The reason for this failure is due to the *tail recursive* form of mult (i.e., there is no context around mult's recursive call). Instead, its result is computed in the *accumulator* parameter z. In the beginning, the accumulator z is initialized with 0. For that reason, we also have this instantiation in our conjecture (3) and in the corresponding induction hypothesis (IH). But as the value of z is changed in each recursive call of while, in the induction conclusion (IC) we have the new value y instead of 0. Thus, the induction conclusion does not correspond to the original conjecture (3) any more and hence, the induction hypothesis (where $z \equiv 0$) cannot be used to prove (IC) (where $z \equiv y$). Hence, tail recursive algorithms like mult are much less suitable for verification tasks than algorithms like times.

The classical solution for this problem is to *generalize* the conjecture (3) to a stronger conjecture which is easier to prove. For instance, in our example one needs the following generalization which can be proved by a suitable induction.

$$\mathsf{mult}(x, y, z) \equiv z + x * y \tag{4}$$

Thus, developing generalization techniques is one of the main challenges in induction theorem proving [Aub79, BM79, HBS92, Wal94, IS97, IB99]. Note that the generalization (4) corresponds to the loop invariant (2) that one would need for a direct verification of the imperative program multiply in the Hoare-calculus. So in fact, finding suitable generalizations is closely related to the search for loop invariants.[2]

[2] A difference between verifying functional programs by induction and verifying imperative programs by loop invariants and inductive assertions is that for imperative programs one uses a "forward" induction starting with the initial values of the program variables and for functional programs a "reversed" induction is used which goes back from their final values to the initial ones. However, the required loop invariants resp. the corresponding generalizations are easily interchangeable, cf. [RY76].

In this paper we propose a new approach to avoid the need for generalizations or loop invariants. The idea is to transform functions like mult, which are difficult to verify, into algorithms like times which are much better amenable to automated induction proofs. For example, the well-known induction theorem proving system NQTHM [BM79, BM98] fails in proving (3), whereas after a transformation of multiply and mult into times this conjecture becomes trivial. This approach of verifying imperative programs via a translation into functional programs is based on the observation that in functional languages there often exists a formulation of the algorithms which is easy to verify (whereas this formulation cannot be expressed in iterative form). The aim of our technique is to find such a formulation automatically.

Our approach has the advantage that the transformation solves the verification problems resulting from a tail recursive algorithm once and for all. On the other hand, when using generalizations or loop invariants one has to find a new generalization (or a new loop invariant, respectively) for every new conjecture about such an algorithm. Moreover, most techniques for finding generalizations or loop invariants have to be guided by the system user, since they rely on the presence of suitable lemmata. By these lemmata the user often has to provide the main idea for the generalization resp. the loop invariant. In contrast, our transformation works automatically.

In particular, automatic generalization techniques fail for many conjectures which contain *several* occurrences of a tail recursive function. As an example, regard the associativity of multiply or, in other words,

$$\mathsf{mult}(\mathsf{mult}(x, y, \mathsf{0}), z, \mathsf{0}) \equiv \mathsf{mult}(x, \mathsf{mult}(y, z, \mathsf{0}), \mathsf{0}). \tag{5}$$

Similar to (3), a direct proof by structural induction on x does not succeed. So again, the standard solution would be to generalize the conjecture (5) by replacing the fixed value $\mathsf{0}$ by suitable terms. For example, one may generalize (5) to

$$\mathsf{mult}(\mathsf{mult}(x, y, \underline{v}), z, \mathsf{0}) \equiv \mathsf{mult}(x, \mathsf{mult}(y, z, \mathsf{0}), \underline{\mathsf{mult}(v, z, \mathsf{0})}).$$

To ease readability, we have underlined those terms where the generalization took place. While the proof of this conjecture is not too hard (using the distributivity of + over multiply), we are not aware of any technique which would find this generalization (or the corresponding loop invariant) automatically, because it is difficult to synthesize the correct replacement of the third argument in the right-hand side (by $\mathsf{mult}(v, z, \mathsf{0})$). The problem is that the disturbing $\mathsf{0}$'s occurring in (5) cannot just be generalized to new variables, since this would yield a flawed conjecture. Thus, finding generalizations for conjectures with several occurrences of a tail recursive function is often particularly hard, as different occurrences of an instantiated accumulator may have to be generalized to different new terms.[3] On the other hand, our transformation allows us to prove such conjectures without user interaction. Essentially, the reason is that while generalizations and loop invariants depend on both the algorithms and the conjectures to be proved, the transformation only depends on the algorithms.

The area of program transformation is a well examined field which has found many applications in software engineering, program synthesis, and compiler construction. For surveys see e.g. [BW82, Par90, MPS93, PP96]. However, the transformations developed for these applications had a goal which is fundamentally different from ours. Our aim is to transform programs into new programs which are easier to verify. In contrast to that, the classical transformation methods aim to increase efficiency. Such transformations are unsuitable for our purpose, since a more efficient algorithm is often harder to verify than a less efficient easier algorithm. Moreover, we want to transform tail recursive algorithms

[3] An alternative generalization of (5) is $\mathsf{mult}(\mathsf{mult}(x, y, \mathsf{0}), z, \underline{v}) \equiv \mathsf{mult}(x, \mathsf{mult}(y, z, \mathsf{0}), \underline{v})$. This generalization is easier to find (as we just replaced both third arguments of the left- and right-hand side by the same new variable v). However, it is not easy to verify (its proof is essentially as hard as the proof of the original conjecture (5)).

into non-tail recursive ones, but in the usual applications of program transformation, non-tail recursive programs are transformed into tail recursive ones ("recursion removal", cf. e.g. [Coo66, DB76, BD77, Wan80, BW82, AK82, HK92]).

As the goals of the existing program transformations are often opposite to ours, a promising approach is to use these classical transformations *in the reverse direction*. To our knowledge, such an application of these transformations for the purpose of verification has rarely been investigated before. In this way, we indeed obtained valuable inspirations for the development of our transformation rules in Section 3 - 5. However, our rules go far beyond the reversed standard program transformation methods, because these methods had to be modified substantially to be applicable for the programs resulting in our context.

3 Context Moving

The only difference between the algorithms mult and times is that the context $\underline{y+}\ \ldots$ to compute the result of times is outside of the recursive call, whereas in mult the context $\underline{y+}\ \ldots$ is in the recursive argument for the accumulator variable z. This change of the accumulator in recursive calls is responsible for the verification problems with mult.

For that reason, we now introduce a transformation rule which allows to *move* the context away from recursive accumulator arguments to a position outside of the recursive call. In this way, the former result $\mathsf{mult}(\mathsf{p}(x), y, \underline{y+}z)$ can be replaced by $\underline{y+}\,\mathsf{mult}(\mathsf{p}(x), y, z)$. So the algorithm mult is transformed into

$$\begin{array}{l}
\textbf{function}\ \mathsf{mult}\,(x, y, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\
\quad \textbf{if}\ x \neq \mathsf{0}\ \textbf{then}\ y\ +\ \mathsf{mult}(\mathsf{p}(x), y, z) \\
\quad \phantom{\textbf{if}\ x \neq \mathsf{0}\ } \textbf{else}\ \ z.
\end{array}$$

To develop a rule for context moving, we have to find sufficient criteria which ensure that such a transformation is equivalence preserving. For our rule, we regard algorithms of the form (6) where the last argument z is used as an accumulator. Our aim is to move the contexts $r_1, \ldots, r_k$ of the recursive accumulator arguments to the top, i.e., to transform the algorithm (6) into (7).

$$\begin{array}{lll}
\textbf{function}\ f\,(x^* : \tau^*, z : \tau) : \tau\ \Leftarrow & & \\
\quad \textbf{if}\ b_1 & \textbf{then}\ f(r_1^*, r_1) & \\
\quad \quad \vdots & & \\
\quad \textbf{if}\ b_k & \textbf{then}\ f(r_k^*, r_k) & (6) \\
\quad \textbf{if}\ b_{k+1} & \textbf{then}\ r_{k+1} & \\
\quad \quad \vdots & & \\
\quad \textbf{if}\ b_m & \textbf{then}\ r_m &
\end{array}
\qquad
\begin{array}{lll}
\textbf{function}\ f\,(x^* : \tau^*, z : \tau) : \tau\ \Leftarrow & & \\
\quad \textbf{if}\ b_1 & \textbf{then}\ r_1[z/f(r_1^*, z)] & \\
\quad \quad \vdots & & \\
\quad \textbf{if}\ b_k & \textbf{then}\ r_k[z/f(r_k^*, z)] & (7) \\
\quad \textbf{if}\ b_{k+1} & \textbf{then}\ r_{k+1} & \\
\quad \quad \vdots & & \\
\quad \textbf{if}\ b_m & \textbf{then}\ r_m. &
\end{array}$$

We demand $m > k \geq 1$, but the order of the f-cases is irrelevant and the transformation may also be applied if the accumulator z is not the *last* parameter of f (we just use the above formulation to ease readability).

First of all, note that the intermediate values of the parameter z are not the same in the two versions of f. Thus, in order to guarantee that evaluation of both versions of f leads to the same cases in the same order, we must demand that the accumulator z does not occur in the conditions $b_1, \ldots, b_m$ or in the recursive arguments $r_1^*, \ldots, r_k^*$ for the other parameters x^*.

Let u^*, w be constructor ground terms. Now for both versions of f, evaluation of $f(u^*, w)$ leads to the same f-cases $i_1, \ldots, i_d$ where $i_1, \ldots, i_{d-1} \in \{1, \ldots, k\}$ and $i_d \in \{k+1, \ldots, m\}$ (provided that the evaluation is defined). Let $t[r^*, s]$ abbreviate $t[x^*/r^*, z/s]$ (where for terms t containing at most the variables x^*, we also write $t[r^*]$) and let $a_h^* = r_{i_h}^*[r_{i_{h-1}}^*[\ldots[r_{i_1}^*[u^*]]\ldots]]$, where $a_0^* = u^*$. Then with the old definition of f we obtain the result (8) and with the new definition we obtain (9).

$$r_{i_d}[a^*_{d-1}, r_{i_{d-1}}[a^*_{d-2}, \ldots r_{i_2}[a^*_1, r_{i_1}[a^*_0, w]] \ldots]] \tag{8}$$
$$r_{i_1}[a^*_0, r_{i_2}[a^*_1, \ldots r_{i_{d-1}}[a^*_{d-2}, r_{i_d}[a^*_{d-1}, w]] \ldots]]. \tag{9}$$

For example, the original algorithm mult computes a result of the form

$$y_x + (y_{x-1} + (\ldots(y_2 + (y_1 + z))\ldots))$$

where y_i denotes the number which is added in the i-th execution of the algorithm. On the other hand, the new version of mult computes the result

$$y_1 + (y_2 + (\ldots(y_{x-1} + (y_x + z))\ldots)).$$

Therefore, the crucial condition for the soundness of this transformation is the *left-commutativity* of the contexts $r_1, \ldots, r_k$ moved, cf. [BW82]. In other words, for all $i \in \{1, \ldots, m\}$ and all $i' \in \{1, \ldots, k\}$ we demand

$$r_i[x^*, r_{i'}[y^*, z]] \equiv r_{i'}[y^*, r_i[x^*, z]].$$

Then (8) and (9) are indeed equal as can be proved by subsequently moving the inner $r_{i_j}[a^*_{j-1}, \ldots]$ contexts of (8) to the top. So for mult, we only have to prove $x+(y+z) \equiv y+(x+z)$ and $y + z \equiv y + z$ (which can easily be verified by the existing induction theorem provers).

Note also that since in the schema (6), $r_1, \ldots, r_m$ denote arbitrary *terms*, such a context moving would also be possible if one would exchange the arguments of $+$ in mult's recursive call. Then r_1 would be $z + y$ and the required left-commutativity conditions would read $(z + y) + x \equiv (z + x) + y$ and $z + y \equiv z + y$.

However, context moving may only be done, if all terms $r_1, \ldots, r_m$ contain the accumulator z. Otherwise f's new definition could be total although the original definition was partial. For example, if f has the (first) case

$$\textbf{if } x \neq 0 \textbf{ then } f(x, 0)$$

then $f(x, z)$ does not terminate for $x \neq 0$. However, if we would not demand that z occurred in the recursive accumulator argument, then context moving could transform this case into "**if** $x \neq 0$ **then** 0". The resulting function is clearly not equivalent to the original one, because now the result of $f(x, z)$ is 0 for $x \neq 0$.

Similarly, z must also occur in the results of non-recursive cases as can be demonstrated with the following example.

function $\mathsf{f}(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow$
if $x \neq 0$ **then** $\mathsf{f}(\mathsf{p}(x), \mathsf{g}(z))$
else 0

function $\mathsf{g}(x : \mathsf{nat}) : \mathsf{nat} \Leftarrow$
if $x \neq 0$ **then** $\mathsf{g}(x)$
else 0

The required left-commutativity conditions are fulfilled and thus, context moving would transform f into

function $\mathsf{f}(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow$
if $x \neq 0$ **then** $\mathsf{g}(\mathsf{f}(\mathsf{p}(x), z))$
else 0.

However, with the original algorithm, $\mathsf{f}(1, 1)$ results in the call $\mathsf{g}(1)$ and hence, it is undefined. On the other hand, in the new algorithm $\mathsf{f}(1, 1)$ is 0.

Finally, we also have to demand that in $r_1, \ldots, r_m$, the accumulator z may not occur within arguments of functions dependent on f. Here, every function is *dependent* on itself and moreover, if g is dependent on f and g occurs in the algorithm h, then h is also dependent on f.

So in particular, this requirement excludes nested recursive calls with the argument z and it also excludes corresponding calls of functions which are mutually recursive with f. Otherwise, the transformation would not preserve the semantics. As an example regard the following function, where the algorithm $\mathsf{one}(z)$ returns 1 for all arguments z.

$$\begin{array}{l} \textbf{function } \mathsf{f}(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if } x \neq \mathsf{0} \textbf{ then } \mathsf{f}(\mathsf{p}(x), \mathsf{f}(z, \mathsf{0})) \\ \qquad\quad \textbf{else } \mathsf{one}(z) \end{array}$$

By moving the context $\mathsf{f}(\ldots, \mathsf{0})$ to the top, the result of the first case would be transformed into $\mathsf{f}(\mathsf{f}(\mathsf{p}(x), z), \mathsf{0})$. The original algorithm satisfies all previously developed conditions. However, the original algorithm is total, whereas after the transformation $\mathsf{f}(x, z)$ does not terminate any more for $x \neq \mathsf{0}$.

Similarly, the occurrence of functions dependent on f in the results $r_{k+1}, \ldots, r_m$ also gives rise to counterexamples.

$$\begin{array}{l} \textbf{function } \mathsf{f}(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if } x \neq \mathsf{0} \textbf{ then } \mathsf{f}(\mathsf{p}(x), \mathsf{p}(z)) \\ \qquad\quad \textbf{else } \mathsf{g}(z) \end{array} \qquad \begin{array}{l} \textbf{function } \mathsf{g}(z : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if } z \neq \mathsf{0} \textbf{ then } \mathsf{f}(z, z) \\ \qquad\quad \textbf{else } \mathsf{0} \end{array}$$

Note that we have $\mathsf{f}(x, z) \equiv \mathsf{0}$ and $\mathsf{g}(z) \equiv \mathsf{0}$ for all numbers x and z. Thus, the required left-commutativity conditions are satisfied and context moving would yield

$$\begin{array}{l} \textbf{function } \mathsf{f}(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if } x \neq \mathsf{0} \textbf{ then } \mathsf{p}(\mathsf{f}(\mathsf{p}(x), z)) \\ \qquad\quad \textbf{else } \mathsf{g}(z). \end{array}$$

However, in contrast to the original version of f, this algorithm is no longer total, since evaluation of $\mathsf{f}(1, 1)$ is not terminating.

Under the above requirements, the transformation of (6) into (7) is sound.[4]

Theorem 1 (Soundness of Context Moving) *Let F be a functional program containing the algorithm (6) and let F' result from F by replacing (6) with (7). Then for all data objects t^*, t, and q, $f(t^*, t)$ evaluates to q in the program F iff it does so in F', provided that the following requirements are fulfilled:*

(A) $z \notin \mathcal{V}(b_1) \cup \ldots \cup \mathcal{V}(b_m)$
(B) $z \notin \mathcal{V}(r_1^*) \cup \ldots \cup \mathcal{V}(r_k^*)$
(C) For all $i \in \{1, \ldots, m\}$, $i' \in \{1, \ldots, k\}$: $F \models_{\mathrm{ind}} r_i[x^*, r_{i'}[y^*, z]] \equiv r_{i'}[y^*, r_i[x^*, z]]$
(D) $z \in \mathcal{V}(r_1) \cap \ldots \cap \mathcal{V}(r_m)$
(E) In $r_1, \ldots, r_m$, *z does not occur in arguments of functions dependent on f.*

In contrast to the original version of mult, the algorithm obtained by context moving is much better suited for verification tasks. The reason is that the (former) accumulator z is initialized with $\mathsf{0}$ and it is no longer changed in the algorithm mult. For that reason, z can now be eliminated from the function mult by replacing all its occurrences by $\mathsf{0}$. The semantics of the main function $\mathsf{multiply}$ remains unchanged by this transformation.

$$\begin{array}{l} \textbf{function } \mathsf{multiply}(x, y : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \mathsf{mult}(x, y) \\ \\ \end{array} \qquad \begin{array}{l} \textbf{function } \mathsf{mult}(x, y : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if } x \neq \mathsf{0} \textbf{ then } y + \mathsf{mult}(\mathsf{p}(x), y) \\ \qquad\quad \textbf{else } \mathsf{0} \end{array}$$

Now mult indeed corresponds to the algorithm times and thus, the complicated generalizations or loop invariants of Section 2 are no longer required. Thus, the verification problems for these procedures are solved.

[4]The proofs for the theorems can be found in Appendix A.

Similarly, context moving can also be applied to transform an algorithm like

$$\begin{array}{lcl} \textbf{function}\ \mathsf{plus}\,(x, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow & & \textbf{function}\ \mathsf{plus}\,(x, y : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \textbf{if}\ x \neq 0\ \textbf{then}\ \mathsf{plus}(\mathsf{p}(x), \mathsf{s}(z)) & \text{into} & \quad \textbf{if}\ x \neq 0\ \textbf{then}\ \mathsf{s}(\mathsf{plus}(\mathsf{p}(x), z)) \\ \qquad\qquad \textbf{else}\ \ z & & \qquad\qquad \textbf{else}\ \ z, \end{array}$$

which is much better suited for verification tasks. Here, for condition (C) we only have to prove $\mathsf{s}(\mathsf{s}(z)) \equiv \mathsf{s}(\mathsf{s}(z))$ and $\mathsf{s}(z) \equiv \mathsf{s}(z)$ (which is trivial).

To apply context moving mechanically, the conditions (A) - (E) for its application have to be checked automatically. For (A), (B), (D), and (E) this is easily done, since these conditions are just syntactic. The left-commutativity condition (C) has to be checked by an underlying induction theorem prover. In many cases, this is not a hard task, since for algorithms like plus the terms $r_i[x^*, r_{i'}[y^*, z]]$ and $r_{i'}[y^*, r_i[x^*, z]]$ are already *syntactically* equal and for algorithms like mult, the required left-commutativity follows from the associativity and commutativity of "+". To ease the proof of such conjectures about auxiliary algorithms, we follow the strategy to apply our transformations to those algorithms first which depend on few other algorithms. Thus, we would attempt to transform "+" before transforming mult. In this way, one can usually avoid the need for generalizations when performing the required left-commutativity proofs. Finally, note that of course, context moving should only be done if at least one of the recursive arguments $r_1, \ldots, r_k$ is different from z (otherwise the algorithm would not change).

Our context moving rule has some similarities to the reversal of a technique known in program transformation (*operand commutation*, cf. e.g. [Coo66,DB76, BW82]). However, our rule generalizes this (reversed) technique substantially.

For example, directly reversing the formulation in [BW82] would result in a rule which would also impose applicability conditions on the functions *that call* the transformed function f (by demanding that f's accumulator would have to be initialized in a certain way). In this way, the applicability of the reversed rule would be unnecessarily restricted (and unnecessarily difficult to check). Therefore, we developed a rule where context moving is separated from the subsequent replacement of the (former) accumulator by initial values like 0. Moreover, in [BW82] the problems concerning the occurrence of the accumulator z and of nested recursive calls are not examined (i.e., the requirements (D) and (E) are missing there). Another important difference is that our rule allows the use of *several different* recursive arguments $r_1, \ldots, r_k$ and the use of *several* non-recursive cases with *arbitrary* results (whereas reversing the formulation in [BW82] would only allow one single recursive case and it would only allow the non-recursive result z instead of the arbitrary terms $r_{k+1}, \ldots, r_m$). Note that for this reason in our rule we have to regard *all* cases of an algorithm at once.

As an example consider the following algorithm to compute the multiplication of all elements in a list, where however occurring 0's are ignored. We use a data type list for lists of naturals with the constructors $\mathsf{nil} : \mathsf{list}$ and $\mathsf{cons} : \mathsf{nat} \times \mathsf{list} \to \mathsf{list}$, where $\mathsf{car} : \mathsf{list} \to \mathsf{nat}$ and $\mathsf{cdr} : \mathsf{list} \to \mathsf{list}$ are the selectors to cons.

$$\begin{array}{l} \textbf{procedure}\ \mathsf{prod}\,(l : \mathsf{list}, z : \mathsf{nat}) \Leftarrow \\ \qquad z := \mathsf{s}(0); \\ \qquad \textbf{while}\ l \neq \mathsf{nil}\ \ \textbf{do}\ \ \textbf{if}\ \mathsf{car}(l) \neq 0\ \textbf{then}\ z := \mathsf{car}(l) * z; \\ \qquad\qquad\qquad\qquad\quad l := \mathsf{cdr}(l) \qquad\qquad\qquad\qquad \textbf{od} \end{array}$$

This procedure can be translated automatically into the following functions (here, we reordered the cases of pr to ease readability).

$$\begin{array}{ll} \textbf{function}\ \ \mathsf{prod}\,(l : \mathsf{list}) : \mathsf{nat} \Leftarrow & \textbf{function}\ \ \mathsf{pr}\,(l : \mathsf{list}, z : \mathsf{nat}) : \mathsf{nat} \Leftarrow \\ \quad \mathsf{pr}(l, \mathsf{s}(0)) & \quad \textbf{if}\ l = \mathsf{nil} \qquad \textbf{then}\ z \\ & \quad \textbf{if}\ \mathsf{car}(l) \neq 0\ \textbf{then}\ \mathsf{pr}(\mathsf{cdr}(l), \mathsf{car}(l) * z) \\ & \qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{pr}(\mathsf{cdr}(l), z) \end{array}$$

To transform the algorithm pr, we indeed need a technique which can handle algorithms with several recursive cases. Since $*$ is left-commutative, context moving and replacing z with $\mathsf{s}(0)$ results in

$$\begin{array}{ll} \textbf{function}\ \mathsf{prod}\,(l:\mathsf{list}):\mathsf{nat} \Leftarrow & \textbf{function}\ \mathsf{pr}\,(l:\mathsf{list}):\mathsf{nat} \Leftarrow \\ \quad \mathsf{pr}(l) & \quad \textbf{if}\ l=\mathsf{nil} \quad \textbf{then}\ \mathsf{s}(0) \\ & \quad \textbf{if}\ \mathsf{car}(l)\neq 0\ \textbf{then}\ \mathsf{car}(l)*\mathsf{pr}(\mathsf{cdr}(l)) \\ & \qquad\qquad\qquad \textbf{else}\ \ \mathsf{pr}(\mathsf{cdr}(l)). \end{array}$$

Further algorithms with several recursive and non-recursive cases where context moving is required are presented in Appendix B.

Moreover, a somewhat related technique was discussed in [Moo75]. However, in contrast to our rule, his transformation is not equivalence-preserving, but it corresponds to a *generalization* of the conjecture. For that reason this approach faces the danger of over-generalization (e.g., the associativity law for multiply is generalized into a flawed conjecture). It turns out that for almost all algorithms considered in [Moo75] our transformation techniques can generate *equivalent* algorithms that are easy to verify. So for such examples, generalizations are no longer needed.

4 Context Splitting

Because of the required left-commutativity, context moving is not always applicable. As an example regard the following imperative procedure for uniting lists. We use a data type llist for lists of list's. Its constructors are empty and add with the selectors hd and tl. So $\mathsf{add}(z,k)$ represents the insertion of the list z in front of the list of lists k and $\mathsf{hd}(\mathsf{add}(z,k))$ yields z. Moreover, we use an algorithm app for list-concatenation. Then after execution of $\mathsf{union}(k,z)$, the value of z is the union of all lists in k.

$$\begin{array}{l} \textbf{procedure}\ \mathsf{union}(k:\mathsf{llist}, z:\mathsf{list}) \Leftarrow \\ \qquad z:=\mathsf{nil}; \\ \qquad \textbf{while}\ k\neq\mathsf{empty}\ \textbf{do}\ z:=\mathsf{app}(\mathsf{hd}(k),z); \\ \qquad\qquad\qquad\qquad\qquad\ k:=\mathsf{tl}(k) \qquad \textbf{od} \end{array}$$

Translation of union into functional algorithms yields

$$\begin{array}{ll} \textbf{function}\ \mathsf{union}\,(k:\mathsf{llist}):\mathsf{list} \Leftarrow & \textbf{function}\ \mathsf{uni}\,(k:\mathsf{llist}, z:\mathsf{list}):\mathsf{list} \Leftarrow \\ \quad \mathsf{uni}(k,\mathsf{nil}) & \quad \textbf{if}\ k\neq\mathsf{empty}\ \textbf{then}\ \mathsf{uni}(\mathsf{tl}(k),\mathsf{app}(\mathsf{hd}(k),z)) \\ & \qquad\qquad\qquad\quad \textbf{else}\ \ z. \end{array}$$

These functions are again unsuited for verification, because the accumulator z of uni is initially called with nil, but this value is changed in the recursive calls. Context moving is not possible, because the context $\mathsf{app}(\mathsf{hd}(k),\ldots)$ is not left-commutative. This motivates the development of the following *context splitting* transformation. Given a list of lists $k = [z_1,\ldots,z_n]$, the result of $\mathsf{uni}(k,\mathsf{nil})$ is

$$\mathsf{app}(z_n,\mathsf{app}(z_{n-1},\ldots\mathsf{app}(z_3,\mathsf{app}(z_2,z_1))\ldots)). \tag{10}$$

In order to move the context of uni's recursive accumulator argument to the top, our aim is to compute this result in a way such that z_1 is moved as far to the "outside" in this term as possible (whereas z_n may be moved to the "inside"). Although app is not commutative, it is at least *associative*. So (10) is equal to

$$\mathsf{app}(\mathsf{app}(\ldots\mathsf{app}(\mathsf{app}(z_n,z_{n-1}),z_{n-2})\ldots,z_2),z_1). \tag{11}$$

This gives an idea on how the algorithm uni may be transformed into a new (unary) algorithm uni′ such that $\mathsf{uni}'(k)$ computes $\mathsf{uni}(k,\mathsf{nil})$. The result of $\mathsf{uni}'([z_1,\ldots,z_n])$ should be $\mathsf{app}(\mathsf{uni}'([z_2, \ldots, z_n]), z_1)$. Similarly, $\mathsf{uni}'([z_2, \ldots, z_n])$ $\mathsf{app}(\mathsf{uni}'([z_3, \ldots, z_n]), z_2)$, etc. Finally, $\mathsf{uni}'([z_n])$ is $\mathsf{app}(\mathsf{uni}'(\mathsf{empty}), z_n)$. To obtain the result (11), $\mathsf{app}(\mathsf{uni}'(\mathsf{empty}), z_n)$ must be equal to z_n. Hence, $\mathsf{uni}'(\mathsf{empty})$ should yield app's neutral argument nil. Thus, we obtain the following new algorithms, which are well suited for verification tasks.

$$
\begin{array}{ll}
\textbf{function}\ \mathsf{union}\,(k : \mathsf{llist}) : \mathsf{list} \ \Leftarrow & \qquad \textbf{function}\ \mathsf{uni}'\,(k : \mathsf{llist}) : \mathsf{list} \ \Leftarrow \\
\quad \mathsf{uni}'(k) & \qquad \quad \textbf{if}\ k \neq \mathsf{empty}\ \textbf{then}\ \mathsf{app}(\mathsf{uni}'(\mathsf{tl}(k)), \mathsf{hd}(k)) \\
 & \qquad \qquad \qquad \quad \textbf{else}\ \ \mathsf{nil}
\end{array}
$$

So the idea is to *split up* the former context $\mathsf{app}(\mathsf{hd}(k), \ldots)$ into an *outer* part $\mathsf{app}(\ldots, \ldots)$ and an *inner* part $\mathsf{hd}(k)$. If the outer context is associative, then one can transform tail recursive results of the form $f(\ldots, \mathsf{app}(\mathsf{hd}(k), z))$ into results of the form $\mathsf{app}(f'(\ldots), \mathsf{hd}(k))$. In general, our context splitting rule generates a new algorithm (13) from an algorithm of the form (12).

$$
\begin{array}{lll}
\textbf{function}\ f\,(x^* : \tau^*, z : \tau) : \tau \ \Leftarrow & & \\
\quad \textbf{if}\ b_1 & \textbf{then}\ f(r_1^*, p[r_1, z]) & \\
 & \vdots & \\
\quad \textbf{if}\ b_k & \textbf{then}\ f(r_k^*, p[r_k, z]) & (12) \\
\quad \textbf{if}\ b_{k+1} & \textbf{then}\ p[r_{k+1}, z] & \\
 & \vdots & \\
\quad \textbf{if}\ b_m & \textbf{then}\ p[r_m, z] &
\end{array}
\qquad
\begin{array}{lll}
\textbf{function}\ f'\,(x^* : \tau^*) : \tau \ \Leftarrow & & \\
\quad \textbf{if}\ b_1 & \textbf{then}\ p[f'(r_1^*), r_1] & \\
 & \vdots & \\
\quad \textbf{if}\ b_k & \textbf{then}\ p[f'(r_k^*), r_k] & (13) \\
\quad \textbf{if}\ b_{k+1} & \textbf{then}\ r_{k+1} & \\
 & \vdots & \\
\quad \textbf{if}\ b_m & \textbf{then}\ r_m. &
\end{array}
$$

Here, p is a term of type τ containing exactly the two new variables x_1 and x_2 of type τ and $p[t_1, t_2]$ abbreviates $p[x_1/t_1, x_2/t_2]$. Then our transformation splits the contexts into their common top part p and their specific part r_i and it moves the common part p to the top of recursive results. (This allows an elimination of the accumulator z.) If there are several possible choices for p, then we use the heuristic to choose p as small and r_i as big as possible. Let e be a constructor ground term which is a neutral argument of p, i.e., $F \models_{\mathrm{ind}} p[x, e] \equiv x$ and $F \models_{\mathrm{ind}} p[e, x] \equiv x$. Then in (12), one may also have z instead of $p[e, z]$. For example, in uni we had the non-recursive result z instead of $\mathsf{app}(\mathsf{nil}, z)$. Moreover we demand $m > k \geq 1$, but the order of the f-cases is again irrelevant and the rule may also be applied if z is not the *last* parameter of f.

We want to ensure that all occurrences of $f(t^*, e)$ in other algorithms g (that f is not dependent on) may be replaced by $f'(t^*)$. For the soundness of this transformation, similar to context moving, the accumulator z must not occur in conditions or in the subterms $r_1^*, \ldots, r_k^*$ or $r_1, \ldots, r_m$. Then for constructor ground terms u^*, the evaluation of $f(u^*, e)$ and of $f'(u^*)$ leads to the same cases $i_1, \ldots, i_d$ where $i_1, \ldots, i_{d-1} \in \{1, \ldots, k\}$ and $i_d \in \{k+1, \ldots, m\}$. For $1 \leq h \leq d$ let a_h be $r_{i_h}[r_{i_{h-1}}^*[\ldots[r_{i_1}^*[u^*]]\ldots]]$. Then the result of $f(u^*, e)$ is (14) and the result of $f'(u^*)$ is (15).

$$p[a_d, p[a_{d-1}, \ldots p[a_2, a_1] \ldots]] \tag{14}$$

$$p[p[\ldots p[p[a_d, a_{d-1}], a_{d-2}] \ldots a_2], a_1] \tag{15}$$

To ensure the equality of these two results, p must be associative. The following theorem summarizes our rule for context splitting.[5]

Theorem 2 (Soundness of Context Splitting) *Let F be a functional program containing (12) and let F' result from F by adding the algorithm (13). Then for all data objects t^* and q, $f(t^*, e)$ evaluates to q in F iff $f'(t^*)$ evaluates to q in F', provided that the following requirements are fulfilled:*

(A) $\quad z \notin \mathcal{V}(b_1) \cup \ldots \cup \mathcal{V}(b_m)$

[5] Again, the proof can be found in Appendix A.

(B) $z \notin \mathcal{V}(r_1^*) \cup \ldots \cup \mathcal{V}(r_k^*) \cup \mathcal{V}(r_1) \cup \ldots \cup \mathcal{V}(r_m)$
(C) $F \models_{\text{ind}} p[p[x_1, x_2], x_3] \equiv p[x_1, p[x_2, x_3]]$
(D) $F \models_{\text{ind}} p[x, e] \equiv x$ *and* $F \models_{\text{ind}} p[e, x] \equiv x$.

Context splitting is only applied if there is a term $f(t^*, e)$ in some other algorithm g that f is not dependent on. In this case, the conditions (C) and (D) are checked by an underlying induction theorem prover (where usually associativity is even easier to prove than (left-)commutativity). Conditions (A) and (B) are just syntactic. In case of success, f' is generated and the term $f(t^*, e)$ in the algorithm g is replaced by $f'(t^*)$.

Similar to context moving, a variant of the above rule if often used in the *reverse* direction (*re-bracketing*, cf. e.g. [Coo66, DB76, BD77, Wan80, BW82, PP96]). Again, instead of directly reversing the technique, we modified and generalized this method, e.g., by regarding several tail recursive and non-tail recursive cases. An algorithm where this general form of our rule is needed will be presented in Section 5 and several others can be found in Appendix B. Moreover, in the next section we will also introduce important refinements which increase the applicability of context splitting considerably and which have no counterpart in the classical re-bracketing rules.

5 Refined Context Splitting

A refinement of our context splitting technique can be used for examples where the context p is not yet in the right form. Regard the following imperative procedure for reversing lists.

procedure $\mathsf{reverse}(l, z : \mathsf{list}) \Leftarrow$
$z := \mathsf{nil}$;
while $l \neq \mathsf{nil}$ **do** $z := \mathsf{cons}(\mathsf{car}(l), z)$;
$l := \mathsf{cdr}(l)$ **od**

By translating reverse into functional form one obtains

function $\mathsf{reverse}(l : \mathsf{list}) : \mathsf{list} \Leftarrow$
$\mathsf{rev}(l, \mathsf{nil})$

function $\mathsf{rev}(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow$
if $l \neq \mathsf{nil}$ **then** $\mathsf{rev}(\mathsf{cdr}(l), \mathsf{cons}(\mathsf{car}(l), z))$
else z.

In order to eliminate the accumulator z, we would like to apply context splitting. Here, the term p in (12) would be $\mathsf{cons}(x_1, x_2)$. But then x_1 would be a variable of type nat (instead of list as required) and hence, the associativity law is not even well typed.

Whenever p has the form $c(x_1, \ldots, x_1, x_2)$ for some constructor c, where x_1 is of the "wrong" type, then one may use the following reformulation of the algorithm. (Of course, here x_2 does not have to be the *last* argument of c.) The idea is to "lift" $x_1, \ldots, x_1$ to an object $\mathsf{lift}_c(x_1, \ldots, x_1)$ of type τ and to define a new function $c' : \tau \times \tau \to \tau$ such that $c'(\mathsf{lift}_c(x_1, \ldots, x_1), x_2) \equiv c(x_1, \ldots, x_1, x_2)$. Moreover, in order to split contexts afterwards, c' should be associative.

As a heuristic, we use the following construction for lift_c and c', provided that apart from c the data type τ just has a constant constructor c_{con}. The function $\mathsf{lift}_c(x_1, \ldots, x_n)$ should yield the term $c(x_1, \ldots, x_n, c_{con})$ and the function c' is defined by the following algorithm (where $d_1, \ldots, d_{n+1}$ are the selectors to c).

function $c'(x, z : \tau) : \tau \;\Leftarrow$
if $x = c(d_1(x), \ldots, d_n(x), d_{n+1}(x))$ **then** $c(d_1(x), \ldots, d_n(x), c'(d_{n+1}(x), z))$
else z

Then $c'(\mathsf{lift}_c(x_1, \ldots, x_n), z) \equiv c(x_1, \ldots, x_n, z)$, c_{con} is a neutral argument for c', and c' is associative. So for rev, we obtain $\mathsf{lift}_{\mathsf{cons}}(x_1) \equiv \mathsf{cons}(x_1, \mathsf{nil})$ and

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{cons}'(x, z : \mathsf{list}) : \mathsf{list}\ \Leftarrow \\
\quad \textbf{if}\ x = \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x))\ \textbf{then}\ \mathsf{cons}(\mathsf{car}(x), \mathsf{cons}'(\mathsf{cdr}(x), z)) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{else}\ \ z.
\end{array}
$$

Note that in this example, cons' corresponds to the concatenation function app.

Thus, the term $\mathsf{cons}(\mathsf{car}(l), z)$ in the algorithm rev may be replaced by $\mathsf{cons}'(\mathsf{lift}_{\mathsf{cons}}(\mathsf{car}(l)), z)$, i.e., by $\mathsf{cons}'(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), z)$. Now the rule for context splitting is applicable which yields

$$
\begin{array}{ll}
\textbf{function}\ \ \mathsf{reverse}(l : \mathsf{list}) : \mathsf{list} \Leftarrow & \textbf{function}\ \ \mathsf{rev}'(l : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \mathsf{rev}'(l) & \quad \textbf{if}\ l \neq \mathsf{nil}\ \textbf{then}\ \mathsf{cons}'(\mathsf{rev}'(\mathsf{cdr}(l)), \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})) \\
 & \qquad\qquad\quad \textbf{else}\ \ \mathsf{nil}.
\end{array}
$$

In contrast to the original versions of $\mathsf{reverse}$ and rev, these algorithms are well suited for verification.

Of course, there are also examples where the context p has the form $g(x_1, x_2)$ for some *algorithm* g (instead of a constructor c) and where x_1 has the "wrong" type. For instance, regard the following imperative procedure to filter all even elements out of a list l. It uses an auxiliary algorithm even and an algorithm $\mathsf{atend}(x, z)$ which inserts an element x at the end of a list z.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atend}(x : \mathsf{nat}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ z = \mathsf{nil}\ \textbf{then}\ \mathsf{cons}(x, \mathsf{nil}) \\
\qquad\qquad\qquad \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(z), \mathsf{atend}(x, \mathsf{cdr}(z)))
\end{array}
$$

Now the procedure filter reads as follows.

$$
\begin{array}{l}
\textbf{procedure}\ \mathsf{filter}(l, z : \mathsf{list}) \Leftarrow \\
\qquad z := \mathsf{nil}; \\
\qquad \textbf{while}\ l \neq \mathsf{nil}\quad \textbf{do}\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ z := \mathsf{atend}(\mathsf{car}(l), z); \\
\qquad\qquad\qquad\qquad\qquad\ \ l := \mathsf{cdr}(l) \qquad\qquad\qquad\qquad\qquad\qquad \textbf{od}
\end{array}
$$

Translating this procedure into functional algorithms yields

$$
\begin{array}{ll}
\textbf{function}\ \ \mathsf{filter}(l : \mathsf{list}) : \mathsf{list} \Leftarrow & \textbf{function}\ \ \mathsf{fil}(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \mathsf{fil}(l, \mathsf{nil}) & \quad \textbf{if}\ l = \mathsf{nil}\qquad\quad \textbf{then}\ z \\
 & \quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{fil}(\mathsf{cdr}(l), \mathsf{atend}(\mathsf{car}(l), z)) \\
 & \qquad\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{fil}(\mathsf{cdr}(l), z).
\end{array}
$$

To apply context splitting for fil, p would be $\mathsf{atend}(x_1, x_2)$ and thus, x_1 would be of type nat instead of list as required. While for constructors like cons, such a problem can be solved by the *lifting* technique described above, now the root of p is the algorithm atend. For such examples, the following *parameter enlargement* transformation often helps.

In the algorithm atend, outside of its own recursive argument the parameter x only occurs in the term $\mathsf{cons}(x, \mathsf{nil})$ and the value of $\mathsf{cons}(x, \mathsf{nil})$ does not change throughout the whole execution of atend (as the value of x does not change in any recursive call). Hence, the parameter x can be "enlarged" into a new parameter y which corresponds to the value of $\mathsf{cons}(x, \mathsf{nil})$. Thus, we result in the following algorithm atend', where $\mathsf{atend}'(\mathsf{cons}(x, \mathsf{nil}), z) \equiv \mathsf{atend}(x, z)$.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atend}'(y, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ z = \mathsf{nil}\ \textbf{then}\ y \\
\qquad\qquad\qquad \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(z), \mathsf{atend}'(y, \mathsf{cdr}(z)))
\end{array}
$$

In general, let $h(x^*, z^*)$ be a function where the parameters x^* are not changed in recursive calls and where x^* only occur within the terms $t_1, \ldots, t_m$ outside of their recursive calls in the algorithm h. If $\mathcal{V}(t_i) \subseteq \{x^*\}$ for all i and if the t_i only contain total functions (like constructors), then one may construct a new algorithm $h'(y_1,\ldots,y_m,z^*)$ by enlarging the parameters x^* into $y_1,\ldots,y_m$. The algorithm h' results from h by replacing all t_i by y_i, where the parameters y_i again remain unchanged in their recursive arguments. Then we have $h'(t_1,\ldots,t_m,z^*) \equiv h(x^*,z^*)$. Thus, in all algorithms f that h is not dependent on, we may replace any subterm $h(s^*,p^*)$ by $h'(t_1[x^*/s^*],\ldots,t_m[x^*/s^*],p^*)$. (The only restriction for this replacement is that all possibly undefined subterms of s^* must still occur in some $t_i[x^*/s^*]$.)

Hence, in the algorithm fil, the term $\mathsf{atend}(\mathsf{car}(l),z)$ can be replaced by $\mathsf{atend}'(\mathsf{cons}(\mathsf{car}(l),\mathsf{nil}),z)$. It turns out that $\mathsf{atend}'(l_1,l_2)$ concatenates the lists l_2 and l_1 (i.e., it corresponds to $\mathsf{app}(l_2,l_1)$). Therefore, atend' is associative and thus, context splitting can be applied to fil now. This yields the following algorithms which are well suited for verification.

```
function filter(l : list) : list ⇐
  fil'(l)

function fil'(l : list) : list ⇐
  if l = nil         then nil
  if even(car(l))    then atend'(fil'(cdr(l)), cons(car(l), nil))
                     else atend'(fil'(cdr(l)), nil)
```

Of course, by subsequent unfolding (or "symbolic evaluation") of atend', the algorithm fil' can be simplified to

```
function fil'(l : list) : list ⇐
  if l = nil         then nil
  if even(car(l))    then cons(car(l), fil'(cdr(l)))
                     else fil'(cdr(l)).
```

Note that here we indeed needed a context splitting rule which can handle algorithms with *several* tail recursive cases. Thus, a direct reversal of the classical re-bracketing rules [BW82] would fail for both reverse and filter (since these rules are restricted to just one recursive case and moreover, they lack the concepts of lifting and of parameter enlargement).

The examples union, reverse, and filter show that context splitting can help in cases where context moving is not applicable. On the other hand for algorithms like plus, context moving is successful, but context splitting is not possible. So none of our two transformations subsumes the other and to obtain a powerful approach, we indeed need *both* of them. But there are also several algorithms where the verification problems can be solved by both context moving and splitting. For example, the algorithms resulting from mult by context moving or splitting only differ in the order of the arguments of $+$ in mult's first recursive case. Thus, both resulting algorithms are well suited for verification tasks.

6 Conclusion

We have presented a new transformational approach for the mechanized verification of imperative programs and tail recursive functions. By our technique, functions that are hard to verify are automatically transformed into functions where verification is significantly easier. Hence, for many programs the invention of loop invariants or of generalizations is no longer required and an automated verification is possible by the existing induction theorem provers. As our transformations generate *equivalent* functions, this transformational verification approach is not restricted to partial correctness, but it can also be used to simplify total correctness and termination proofs [Gie95, Gie97, GWB98, AG99, BG99]. See Appendix B for a collection of examples that demonstrates the power of our approach. It shows that our transformation indeed simplifies the verification tasks substantially for many practically relevant algorithms from different areas of computer science (e.g., arithmetical algorithms or procedures for processing (possibly multidimensional) lists including algorithms for matrix multiplication and sorting algorithms like selection-, insertion-, and merge-sort, etc.). Based on the rules and heuristics presented, we implemented a system to perform such transformations automatically [Gie99a].

The field of mechanized verification and induction theorem proving represents a new application area for program transformation techniques. It turns out that our approach of transforming algorithms often seems to be superior to the classical solution of generalizing theorems. For instance, our technique automatically transforms all (first order) tail recursive functions treated in recent generalization techniques [IS97, IB99] into non-tail recursive ones whose verification is very simple. On the other hand, the techniques for finding generalizations are mostly semi-automatic (since they are guided by the system user who has to provide suitable lemmata). Obviously, by formulating the right lemmata (interactively), in principle generalization techniques can deal with almost every conjecture to be proved. But in particular for conjectures which involve *several* occurrences of a tail recursive function, finding suitable generalizations is often impossible for fully automatic techniques. Therefore, our approach represents a significant contribution for mechanized verification of imperative and tail recursive functional programs. Nevertheless, of course there also exist tail recursive algorithms where our automatic transformations are not applicable. For such examples, (interactive) generalizations are still required.

Further work will include an examination of other existing program transformation techniques in order to determine whether they can be modified into transformations suitable for an application in the program verification domain. Moreover, in future work the application area of program verification may also give rise to new transformations which have no counterpart at all in classical program transformations.

A Proofs

Theorem 1 (Soundness of Context Moving) *Let F be a functional program containing the algorithm (6) and let F' result from F by replacing (6) with (7). Then for all data objects t^*, t, and q, $f(t^*, t)$ evaluates to q in the program F iff it does so in F', provided that the following requirements are fulfilled:*

(A) $z \notin \mathcal{V}(b_1) \cup \ldots \cup \mathcal{V}(b_m)$
(B) $z \notin \mathcal{V}(r_1^*) \cup \ldots \cup \mathcal{V}(r_k^*)$
(C) For all $i \in \{1, \ldots, m\}$, $i' \in \{1, \ldots, k\}$: $F \models_{\text{ind}} r_i[x^*, r_{i'}[y^*, z]] \equiv r_{i'}[y^*, r_i[x^*, z]]$
(D) $z \in \mathcal{V}(r_1) \cap \ldots \cap \mathcal{V}(r_m)$
(E) In $r_1, \ldots, r_m$, *z does not occur in arguments of functions dependent on f.*

Proof. For the "if"-direction, we first prove the following context moving lemma for all constructor ground terms u^*, v^*, w, q and all $i' \in \{1, \ldots, k\}$:

$$\text{If } F \models_{\text{ind}} r_{i'}[v^*, f(u^*, w)] \equiv q, \text{ then } F \models_{\text{ind}} f(u^*, r_{i'}[v^*, w]) \equiv q. \qquad (16)$$

We use an induction on the length of $r_{i'}[v^*, f(u^*, w)]$'s evaluation. Due to Condition (D), we have $z \in \mathcal{V}(r_{i'})$ and thus, evaluation of $f(u^*, w)$ is defined as well. Hence, there is an $i \in \{1, \ldots, m\}$ such that $b_i[u^*] \equiv_F \mathsf{true}$ and $b_j[u^*] \equiv_F \mathsf{false}$ for all $1 \leq j < i$, where $s \equiv_F t$ abbreviates $F \models_{\text{ind}} s \equiv t$. If $i \geq k+1$, then

$$\begin{aligned} r_{i'}[v^*, f(u^*, w)] &\equiv_F r_{i'}[v^*, r_i[u^*, w]] \\ &\equiv_F r_i[u^*, r_{i'}[v^*, w]], && \text{by (C)} \\ &\equiv_F f(u^*, r_{i'}[v^*, w]), && \text{since } z \in \mathcal{V}(r_i) \text{ (by (D)).} \end{aligned}$$

If $i \leq k$, then we have

$$\begin{aligned} r_{i'}[v^*, f(u^*, w)] &\equiv_F r_{i'}[v^*, f(r_i^*[u^*], r_i[u^*, w])] \\ &\equiv_F f(r_i^*[u^*], r_{i'}[v^*, r_i[u^*, w]]), && \text{by the induction hypothesis} \\ &\equiv_F f(r_i^*[u^*], r_i[u^*, r_{i'}[v^*, w]]), && \text{by (C)} \\ &\equiv_F f(u^*, r_{i'}[v^*, w]), && \text{since } z \in \mathcal{V}(r_i) \text{ (by (D)).} \end{aligned}$$

Thus, Lemma (16) is proved and now the "if"-direction of Thm. 1 can be shown by induction on the length of $f(t^*,t)$'s evaluation in F'. There must be an $i' \in \{1,\ldots,m\}$ such that $b_{i'}[t^*] \equiv_{F'} \mathsf{true}$ and $b_{j'}[t^*] \equiv_{F'} \mathsf{false}$ for all $1 \le j' < i'$. The induction hypothesis implies $b_{i'}[t^*] \equiv_F \mathsf{true}$ and $b_{j'}[t^*] \equiv_F \mathsf{false}$ as well.

If $i' \ge k+1$, then the conjecture follows from $f(t^*,t) \equiv_{F'} r_{i'}[t^*,t]$, $f(t^*,t) \equiv_F r_{i'}[t^*,t]$, and the induction hypothesis. If $i' \le k$, then we have $f(t^*,t) \equiv_{F'} r_{i'}[t^*, f(r^*_{i'}[t^*],t)] \equiv_{F'} q$ for some constructor ground term q. By the induction hypothesis we obtain $r_{i'}[t^*, f(r^*_{i'}[t^*],t)] \equiv_F q$ and Lemma (16) implies $f(r^*_{i'}[t^*], r_{i'}[t^*,t]) \equiv_F r_{i'}[t^*, f(r^*_{i'}[t^*],t)]$. As $f(t^*,t) \equiv_F f(r^*_{i'}[t^*], r_{i'}[t^*,t])$, the "if"-direction of Thm. 1 is proved.

The proof for the "only if"-direction has a similar structure, but instead of an induction on the length of the evaluation, we need an induction w.r.t. the relation $\succ_f$. Here, $u^* \succ_f q^*$ holds for the constructor ground terms u^* and q^* iff there exist constructor ground terms u and q such that $f(u^*,u)$ is defined in F and such that F-evaluation of $f(u^*,u)$ leads to the recursive call $f(q^*,q)$. The reason for this asymmetry in the proof is that the left-commutativity condition (C) is only demanded for the original program F.

Note that by the requirements (A), (B), and (E), $u^* \succ_f q^*$ implies that for *all* constructor ground terms u where $f(u^*,u)$ is defined, there exists a constructor ground term q such that evaluation of $f(u^*,u)$ leads to evaluation of $f(q^*,q)$. Hence, $\succ_f$ is well founded (i.e., it may indeed be used for induction proofs). Now the reverse direction of Lemma (16) can be proved by induction w.r.t. $\succ_f$.

$$\text{If } F \models_{\text{ind}} f(u^*, r_{i'}[v^*,w]) \equiv q, \text{ then } F \models_{\text{ind}} r_{i'}[v^*, f(u^*,w)] \equiv q. \tag{17}$$

The proof of (17) is analogous to the one of (16), but if evaluation of $f(u^*, r_{i'}[v^*,w])$ leads to execution of a case i with $i \le k$, then we need the induction hypothesis to infer $f(r^*_i[u^*], r_{i'}[v^*, r_i[u^*,w]]) \equiv_F r_{i'}[v^*, f(r^*_i[u^*], r_i[u^*,w])]$. This would not be possible if we performed induction on the length of the evaluation, but it can be done with our induction relation, since $r^*_i[u^*] \equiv_F q^*$ for some constructor ground terms q^* with $u^* \succ_f q^*$.

Finally, the "only if"-direction of the theorem is also proved by induction w.r.t. $\succ_f$. If F-evaluation of $f(t^*,t)$ leads to the i'-th case and $i' \ge k+1$, then the proof is analogous to the "if"-direction. If $i' \le k$, then we have $f(t^*,t) \equiv_F f(r^*_{i'}[t^*], r_{i'}[t^*,t]) \equiv_F r_{i'}[t^*, f(r^*_{i'}[t^*],t)]$ by Lemma (17). Note that for all f-subterms $f(s^*,s)$ in this term, s^* evaluates to constructor ground terms q^* with $t^* \succ_f q^*$. For f-subterms where the root is in $r_{i'}$, this follows from Condition (E). (However, the length of the evaluation $r_{i'}[t^*, f(r^*_{i'}[t^*],t)]$ is not necessarily smaller than the one of $f(t^*,t)$, i.e., we really need an induction w.r.t. $\succ_f$.)

Then by the induction hypothesis, $r_{i'}[t^*, f(r^*_{i'}[t^*],t)] \equiv_F q$ for some constructor ground term q implies $r_{i'}[t^*, f(r^*_{i'}[t^*],t)] \equiv_{F'} q$ and hence, $f(t^*,t) \equiv_{F'} q$. $\square$

Theorem 2 (Soundness of Context Splitting) *Let F be a functional program containing (12) and let F' result from F by adding the algorithm (13). Then for all data objects t^* and q, $f(t^*,e)$ evaluates to q in F iff $f'(t^*)$ evaluates to q in F', provided that the following requirements are fulfilled:*

(A) $z \notin \mathcal{V}(b_1) \cup \ldots \cup \mathcal{V}(b_m)$
(B) $z \notin \mathcal{V}(r^*_1) \cup \ldots \cup \mathcal{V}(r^*_k) \cup \mathcal{V}(r_1) \cup \ldots \cup \mathcal{V}(r_m)$
(C) $F \models_{\text{ind}} p[p[x_1,x_2],x_3] \equiv p[x_1,p[x_2,x_3]]$
(D) $F \models_{\text{ind}} p[x,e] \equiv x$ *and* $F \models_{\text{ind}} p[e,x] \equiv x$.

Proof. Note that evaluation of f is the same in F and F'. Moreover, Conditions (C) and (D) also hold for F'. We prove the (stronger) conjecture

$$f(t^*,t) \equiv_{F'} q \;\text{ iff }\; p[f'(t^*),t] \equiv_{F'} q \tag{18}$$

for all constructor ground terms t^*, t, and q.

The "only if"-direction of (18) is proved by induction on the length of $f(t^*, t)$'s evaluation. There must be an $i \in \{1, \ldots, m\}$ such that $b_i[t^*] \equiv_{F'} \mathsf{true}$ and $b_j[t^*] \equiv_{F'} \mathsf{false}$ for all $1 \leq j < i$. If $i \geq k+1$, then we have

$$f(t^*, t) \equiv_{F'} p[r_i[t^*], t] \equiv_{F'} p[f'(t^*)t].$$

If $i \leq k$, then we obtain

$$\begin{array}{rcll} f(t^*, t) & \equiv_{F'} & f(r_i^*[t^*], p[r_i[t^*], t]) & \\ & \equiv_{F'} & p[f'(r_i^*[t^*]), p[r_i[t^*], t]], & \text{by the induction hypothesis} \\ & \equiv_{F'} & p[p[f'(r_i^*[t^*]), r_i[t^*]], t], & \text{by (C)} \\ & \equiv_{F'} & p[f'(t^*), t]. & \end{array}$$

For the "if"-direction of (18) we use an induction w.r.t. the relation $\succ_{f'}$, where $u^* \succ_{f'} q^*$ holds for two tuples of constructor ground terms u^* and q^* iff evaluation of $f'(u^*)$ is defined and it leads to evaluation of $f'(q^*)$.

As $x_1 \in \mathcal{V}(p)$, evaluation of $f'(t^*)$ is defined and thus, it results in execution of some case i. Now the proof is analogous to the "only if"-direction. (Note that if $i \leq k$, to conclude $p[f'(r_i^*[t^*]), p[r_i[t^*], t]] \equiv_{F'} f(r_i^*[t^*], p[r_i[t^*], t])$, we really need an induction w.r.t. $\succ_{f'}$, whereas an induction on the length of the evaluation does not work.) $\square$

B Examples

This section contains a collection of 55 tail recursive algorithms where context moving or context splitting can be applied in order to transform them into algorithms which are better suited for (possibly mechanized) verification.

B.1 plus

This algorithm adds two numbers.

$$\begin{array}{l} \textbf{function}\ \ \mathsf{plus}\,(x, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x = \mathsf{0}\ \textbf{then}\ \mathsf{plus}(\mathsf{p}(x), \mathsf{s}(z)) \\ \qquad\qquad\ \textbf{else}\ \ z \end{array}$$

Context moving and replacing z with $\mathsf{0}$ results in

$$\begin{array}{l} \textbf{function}\ \ \mathsf{plus}\,(x, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x = \mathsf{0}\ \textbf{then}\ \mathsf{s}(\mathsf{plus}(\mathsf{p}(x), z)) \\ \qquad\qquad\ \textbf{else}\ \ z. \end{array}$$

In the following, we often abbreviate plus with $+$.

B.2 multiply

The following tail recursive multiplication algorithm was used to introduce the technique of context moving in the paper.

$$\textbf{function}\ \ \mathsf{multiply}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{mult}(x, y, \mathsf{0})$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{mult}\,(x, y, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x \neq \mathsf{0}\ \textbf{then}\ \mathsf{mult}(\mathsf{p}(x), y, y + z) \\ \qquad\qquad\ \textbf{else}\ \ z \end{array}$$

As $+$ is left-commutative, we can apply context moving. Subsequently, all occurrences of z in the algorithm mult can be replaced by 0.

$$
\begin{array}{l}
\textbf{function } \ \mathsf{multiply}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \ \mathsf{mult}(x, y) \\[1em]
\textbf{function } \ \mathsf{mult}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if } x \neq 0 \textbf{ then } y + \mathsf{mult}(\mathsf{p}(x), y) \\
\qquad\qquad\ \ \textbf{else } \ 0
\end{array}
$$

Of course, in this and all other corresponding examples, one may also exchange cases, exchange the parameters x and y, and (resp. or) exchange the arguments of the left-commutative function ("$+$" in the above example). The corresponding transformation by context moving would still be possible. In the following, we often abbreviate multiplication algorithms like times or $\mathsf{multiply}$ with $*$.

Note that a suitable transformation of $\mathsf{multiply}$ would also be possible by context splitting. This would yield

$$
\begin{array}{l}
\textbf{function } \ \mathsf{multiply}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \ \mathsf{mult}'(x, y) \\[1em]
\textbf{function } \ \mathsf{mult}'\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if } x \neq 0 \textbf{ then } \mathsf{mult}'(\mathsf{p}(x), y) + y \\
\qquad\qquad\ \ \textbf{else } \ 0.
\end{array}
$$

Both resulting versions of $\mathsf{multiply}$ are well suited for verification tasks. Similarly, in many of the following examples (where we have a *binary* left-commutative auxiliary function whose both arguments are of the same type), instead of context moving one could also use context splitting.

B.3 multiply2

This algorithm also computes multiplication, but in contrast to $\mathsf{multiply}$ it does not use an auxiliary algorithm for addition. Instead, the addition is encoded into the multiplication algorithm as well.

$$
\begin{array}{l}
\textbf{function } \ \mathsf{multiply2}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \ \mathsf{mult2}(x, y, y, 0) \\[1em]
\textbf{function } \ \mathsf{mult2}\,(x, y, r, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if } x = 0 \textbf{ then } z \\
\quad \textbf{if } r = 0 \textbf{ then } \mathsf{mult2}(\mathsf{p}(x), y, y, z) \\
\qquad\qquad\ \ \textbf{else } \ \mathsf{mult2}(x, y, \mathsf{p}(r), \mathsf{s}(z))
\end{array}
$$

Note that this algorithm requires the use of a transformation rule which can handle several recursive cases. Context moving and replacement of the parameter z by 0 yields

$$
\begin{array}{l}
\textbf{function } \ \mathsf{multiply2}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \ \mathsf{mult2}(x, y, y) \\[1em]
\textbf{function } \ \mathsf{mult2}\,(x, y, r : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if } x = 0 \textbf{ then } 0 \\
\quad \textbf{if } r = 0 \textbf{ then } \mathsf{mult2}(\mathsf{p}(x), y, y) \\
\qquad\qquad\ \ \textbf{else } \ \mathsf{s}(\mathsf{mult2}(x, y, \mathsf{p}(r))).
\end{array}
$$

B.4 double

The next algorithm duplicates natural numbers.

$$\textbf{function}\ \ \mathsf{double}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{do}(x, 0)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{do}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{do}(\mathsf{p}(x), \mathsf{s}(\mathsf{s}(z))) \end{array}$$

Here, context moving (and replacing z with 0) results in

$$\textbf{function}\ \ \mathsf{double}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{do}(x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{do}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ 0 \\ \qquad\qquad \textbf{else}\ \ \mathsf{s}(\mathsf{s}(\mathsf{do}(\mathsf{p}(x)))). \end{array}$$

B.5 half

The next algorithm halves natural numbers, i.e., $\mathsf{half}(x)$ computes $\lfloor \frac{x}{2} \rfloor$.

$$\textbf{function}\ \ \mathsf{half}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{ha}(x, 0)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{ha}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0 \quad\ \ \textbf{then}\ z \\ \quad \textbf{if}\ x = \mathsf{s}(0)\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{ha}(\mathsf{p}(\mathsf{p}(x)), \mathsf{s}(z)) \end{array}$$

Context moving (and replacing z with 0) results in

$$\textbf{function}\ \ \mathsf{half}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{ha}(x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{ha}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0 \quad\ \ \textbf{then}\ 0 \\ \quad \textbf{if}\ x = \mathsf{s}(0)\ \textbf{then}\ 0 \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{s}(\mathsf{ha}(\mathsf{p}(\mathsf{p}(x)))). \end{array}$$

B.6 half2

Similarly, the following algorithm also halves natural numbers, but it works from "top to bottom".

$$\textbf{function}\ \ \mathsf{half2}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{ha2}(x, x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{ha2}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0 \quad\ \ \textbf{then}\ z \\ \quad \textbf{if}\ x = \mathsf{s}(0)\ \textbf{then}\ \mathsf{p}(z) \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{ha2}(\mathsf{p}(\mathsf{p}(x)), \mathsf{p}(z)) \end{array}$$

Note that here we need our context moving rule which can deal with several different non-recursive results. In this way, we result in

$$\textbf{function}\ \ \mathsf{half2}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{ha2}(x, x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{ha2}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0 \quad\ \ \textbf{then}\ z \\ \quad \textbf{if}\ x = \mathsf{s}(0)\ \textbf{then}\ \mathsf{p}(z) \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{p}(\mathsf{ha2}(\mathsf{p}(\mathsf{p}(x)), z)). \end{array}$$

B.7 multiply3

The following is again a multiplication algorithm, but this time even numbers are treated differently from odd ones.

$$\begin{array}{l}
\textbf{function}\ \ \mathsf{multiply3}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{mult3}(x, y, 0) \\[1ex]
\textbf{function}\ \ \mathsf{mult3}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x = 0 \quad \textbf{then}\ z \\
\quad \textbf{if}\ \mathsf{even}(x)\ \textbf{then}\ \mathsf{mult3}(\mathsf{half}(x), \mathsf{double}(y), z) \\
\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{mult3}(\mathsf{p}(x), y, y + z)
\end{array}$$

Note that this algorithm again requires a transformation rule that can deal with several recursive cases. Our context moving rule yields the following algorithms (where we replaced all occurrences of z by 0 again).

$$\begin{array}{l}
\textbf{function}\ \ \mathsf{multiply3}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{mult3}(x, y) \\[1ex]
\textbf{function}\ \ \mathsf{mult3}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x = 0 \quad \textbf{then}\ 0 \\
\quad \textbf{if}\ \mathsf{even}(x)\ \textbf{then}\ \mathsf{mult3}(\mathsf{half}(x), \mathsf{double}(y)) \\
\qquad\qquad\quad\ \ \textbf{else}\ \ y + \mathsf{mult3}(\mathsf{p}(x), y)
\end{array}$$

B.8 multiply_succ

The following algorithm computes $x * y + 1$.

$$\begin{array}{l}
\textbf{function}\ \ \mathsf{multiply_succ}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{multsucc}(x, y, 0) \\[1ex]
\textbf{function}\ \ \mathsf{multsucc}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x \neq 0\ \textbf{then}\ \mathsf{multsucc}(\mathsf{p}(x), y, y + z) \\
\qquad\qquad\ \ \textbf{else}\ \ \mathsf{s}(z)
\end{array}$$

Although the non-recursive result has a different context s than the recursive accumulator argument, we can still apply context moving. For the desired left-commutative cooperation of r_2 and r_1 one has to prove

$$\mathsf{s}(y + z) \equiv y + \mathsf{s}(z),$$

which is in fact true. Subsequently, all occurrences of z in the algorithm multsucc can be replaced by 0.

$$\begin{array}{l}
\textbf{function}\ \ \mathsf{multiply_succ}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{multsucc}(x, y) \\[1ex]
\textbf{function}\ \ \mathsf{multsucc}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x \neq 0\ \textbf{then}\ y + \mathsf{multsucc}(\mathsf{p}(x), y) \\
\qquad\qquad\ \ \textbf{else}\ \ \mathsf{s}(0)
\end{array}$$

B.9 minus

Similar to half and half2 one can also transform corresponding subtraction algorithms. Here, we use an auxiliary algorithm $>$ for the usual greater-relation on naturals.

$$\begin{array}{l}
\textbf{function}\ \ \mathsf{minus}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{mi}(x, y, 0) \\[1ex]
\textbf{function}\ \ \mathsf{mi}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\
\quad \textbf{if}\ x > y\ \textbf{then}\ \mathsf{mi}(\mathsf{p}(x), y, \mathsf{s}(z)) \\
\qquad\qquad\ \ \textbf{else}\ \ z.
\end{array}$$

Context moving (and replacing z with 0) results in

$$\textbf{function}\ \ \mathsf{minus}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{mi}(x, y)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{mi}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x > y\ \textbf{then}\ \mathsf{s}(\mathsf{mi}(\mathsf{p}(x), y)) \\ \qquad\qquad\ \ \textbf{else}\ \ \mathsf{0}.\end{array}$$

B.10 minus2

Analogously, this alternative subtraction algorithm works from "top to bottom".

$$\begin{array}{l}\textbf{function}\ \ \mathsf{minus2}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ y = \mathsf{0}\ \textbf{then}\ x \\ \qquad\qquad\ \ \textbf{else}\ \ \mathsf{minus2}(\mathsf{p}(x), \mathsf{p}(y))\end{array}$$

As p is also left-commutative, context moving yields

$$\begin{array}{l}\textbf{function}\ \ \mathsf{minus2}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ y = \mathsf{0}\ \textbf{then}\ x \\ \qquad\qquad\ \ \textbf{else}\ \ \mathsf{p}(\mathsf{minus2}(x, \mathsf{p}(y))).\end{array}$$

B.11 logarithm

The following algorithm computes the truncated logarithm w.r.t. base 2 using an auxiliary algorithm $>$.

$$\textbf{function}\ \ \mathsf{logarithm}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{log}(x, \mathsf{0})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{log}\,(x, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x > \mathsf{s}(\mathsf{0})\ \textbf{then}\ \mathsf{log}(\mathsf{half}(x), \mathsf{s}(z)) \\ \qquad\qquad\qquad\ \ \textbf{else}\ \ z\end{array}$$

Context moving and replacement of the parameter z by 0 yields

$$\textbf{function}\ \ \mathsf{logarithm}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{log}(x)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{log}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x > \mathsf{s}(\mathsf{0})\ \textbf{then}\ \mathsf{s}(\mathsf{log}(\mathsf{half}(x))) \\ \qquad\qquad\qquad\ \ \textbf{else}\ \ \mathsf{0}.\end{array}$$

B.12 power_two

The next function computes the greatest power of 2 that is less than or equal to x.

$$\textbf{function}\ \ \mathsf{power\_two}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{pt}(x, \mathsf{s}(\mathsf{0}))$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{pt}\,(x, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x > \mathsf{s}(\mathsf{0})\ \textbf{then}\ \mathsf{pt}(\mathsf{half}(x), \mathsf{double}(z)) \\ \qquad\qquad\qquad\ \ \textbf{else}\ \ z\end{array}$$

Context moving can be used to move the auxiliary function double to the outside. Afterwards, a replacement of the parameter z by s(0) yields

$$\textbf{function}\ \ \mathsf{power\_two}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{pt}(x)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{pt}\,(x : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x > \mathsf{s}(\mathsf{0})\ \textbf{then}\ \mathsf{double}(\mathsf{pt}(\mathsf{half}(x))) \\ \qquad\qquad\qquad\ \ \textbf{else}\ \ \mathsf{s}(\mathsf{0}).\end{array}$$

B.13 quotient

The following algorithm computes the truncated division $\lfloor \frac{x}{y} \rfloor$ using an algorithm $\geq$ for comparing natural numbers. Here, "$-$" abbreviates the algorithm minus.

$$\textbf{function}\ \ \mathsf{quotient}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{quot}(x, y, \mathsf{0})$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{quot}\,(x, y, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x \geq y\ \textbf{then}\ \mathsf{quot}(x - y, y, \mathsf{s}(z)) \\ \qquad\qquad\quad \textbf{else}\ \ z \end{array}$$

Context moving and replacement of the parameter z by $\mathsf{0}$ yields

$$\textbf{function}\ \ \mathsf{quotient}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{quot}(x, y)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{quot}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x \geq y\ \textbf{then}\ \mathsf{s}(\mathsf{quot}(x - y, y)) \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{0}. \end{array}$$

B.14 quotient2

This algorithm also computes truncated division, but in contrast to $\mathsf{quotient}$ this time the subtraction is encoded into the division algorithm as well (this is similar to the encoding of addition into the multiplication algorithm $\mathsf{multiply2}$ in Example B.3). The auxiliary boolean algorithm and for conjunction is abbreviated by $\wedge$.

$$\textbf{function}\ \ \mathsf{quotient2}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{quot2}(x, y, y, \mathsf{0})$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{quot2}\,(x, y, r, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x = \mathsf{0} \wedge r = \mathsf{0}\ \textbf{then}\ \mathsf{s}(z) \\ \quad \textbf{if}\ x = \mathsf{0} \wedge r \neq \mathsf{0}\ \textbf{then}\ z \\ \quad \textbf{if}\qquad\quad\ r = \mathsf{0}\ \textbf{then}\ \mathsf{quot2}(x, y, y, \mathsf{s}(z)) \\ \qquad\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{quot2}(\mathsf{p}(x), y, \mathsf{p}(r), z) \end{array}$$

Note that this algorithm requires the use of a transformation rule which can handle several recursive and non-recursive cases. Context moving and replacement of the parameter z by $\mathsf{0}$ yields

$$\textbf{function}\ \ \mathsf{quotient2}\,(x, y : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow\ \mathsf{quot2}(x, y, y)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{quot2}\,(x, y, r : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ x = \mathsf{0} \wedge r = \mathsf{0}\ \textbf{then}\ \mathsf{s}(\mathsf{0}) \\ \quad \textbf{if}\ x = \mathsf{0} \wedge r \neq \mathsf{0}\ \textbf{then}\ \mathsf{0} \\ \quad \textbf{if}\qquad\quad\ r = \mathsf{0}\ \textbf{then}\ \mathsf{s}(\mathsf{quot2}(x, y, y)) \\ \qquad\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{quot2}(\mathsf{p}(x), y, \mathsf{p}(r)). \end{array}$$

B.15 length

This algorithm computes the length of a list.

$$\textbf{function}\ \ \mathsf{length}\,(l : \mathsf{list}) : \mathsf{nat}\ \Leftarrow\ \mathsf{len}(l, \mathsf{0})$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{len}\,(l : \mathsf{list}, z : \mathsf{nat}) : \mathsf{nat}\ \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{len}(\mathsf{cdr}(l), \mathsf{s}(z)) \end{array}$$

Context moving and replacing z with $\mathsf{0}$ results in

$$\textbf{function}\ \ \mathsf{length}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{len}(l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{len}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{0}\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{s}(\mathsf{len}(\mathsf{cdr}(l))).\end{array}$$

B.16 index

The function $\mathsf{index}(x,l)$ computes the first index z, such that the z-th element of l is x. Here, the leftmost element of l has index 0. We again use an auxiliary algorithm and for conjunction (which we abbreviate by $\wedge$).

$$\textbf{function}\ \ \mathsf{index}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{ind}(x,l,\mathsf{0})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ind}\,(x:\mathsf{nat},l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l\neq\mathsf{nil}\ \wedge\ \mathsf{car}(l)\neq x\ \textbf{then}\ \mathsf{ind}(x,\mathsf{cdr}(l),\mathsf{s}(z))\\ \qquad\qquad\qquad\qquad\qquad\quad \textbf{else}\ \ z\end{array}$$

Context moving and replacing z with $\mathsf{0}$ results in

$$\textbf{function}\ \ \mathsf{index}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{ind}(x,l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ind}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l\neq\mathsf{nil}\ \wedge\ \mathsf{car}(l)\neq x\ \textbf{then}\ \mathsf{s}(\mathsf{ind}(x,\mathsf{cdr}(l)))\\ \qquad\qquad\qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{0}.\end{array}$$

B.17 sum

The function sum computes the sum of all elements of a list.

$$\textbf{function}\ \ \mathsf{sum}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{su}(l,\mathsf{0})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{su}\,(l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{su}(\mathsf{cdr}(l),\mathsf{car}(l)+z)\end{array}$$

By the left-commutativity of $+$, context moving and replacing z with $\mathsf{0}$ results in

$$\textbf{function}\ \ \mathsf{sum}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{su}(l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{su}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{0}\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{car}(l)+\mathsf{su}(\mathsf{cdr}(l)).\end{array}$$

B.18 weight

Similar to the previous algorithm, the following algorithm computes the *weighted* sum of the elements in a list. In other words, we have $\mathsf{weight}([a_0,\ldots,a_n])=\sum_{i=0,\ldots,n} i*a_i$.

$$\textbf{function}\ \ \mathsf{weight}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{we}(l,\mathsf{0},\mathsf{0})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{we}\,(l:\mathsf{list},i,z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{we}(\mathsf{cdr}(l),\mathsf{s}(i),i*\mathsf{car}(l)+z)\end{array}$$

Again, context moving and replacing z with $\mathsf{0}$ results in

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{we}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{we}(l,\mathsf{0})\\[1em]
\textbf{function}\ \ \mathsf{we}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\\
\quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{0}\\
\qquad\qquad\quad\ \ \textbf{else}\ \ i*\mathsf{car}(l)+\mathsf{we}(\mathsf{cdr}(l),\mathsf{s}(i)).
\end{array}
$$

B.19 count_even

This algorithm counts the number of even elements in a list (using an auxiliary algorithm even).

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{count\_even}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{ce}(l,\mathsf{0})\\[1em]
\textbf{function}\ \ \mathsf{ce}\,(l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\
\quad \textbf{if}\ l=\mathsf{nil}\qquad\quad\ \textbf{then}\ z\\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{ce}(\mathsf{cdr}(l),\mathsf{s}(z))\\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{ce}(\mathsf{cdr}(l),z)
\end{array}
$$

To transform this algorithm, we need a technique which can handle algorithms with several recursive cases. Context moving and replacing z with $\mathsf{0}$ results in

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{count\_even}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{ce}(l)\\[1em]
\textbf{function}\ \ \mathsf{ce}\,(l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\
\quad \textbf{if}\ l=\mathsf{nil}\qquad\quad\ \textbf{then}\ \mathsf{0}\\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{s}(\mathsf{ce}(\mathsf{cdr}(l)))\\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{ce}(\mathsf{cdr}(l)).
\end{array}
$$

Analogously, context moving would also work for similar algorithms (i.e., a "count" algorithm where even elements are counted twice whereas odd ones are just counted once, or a "sum" algorithm which only adds the even elements of a list).

B.20 prod

This algorithm (from Section 3) computes the multiplication of all elements in a list, where however occurring $\mathsf{0}$'s are ignored.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{prod}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{pr}(l,\mathsf{s}(\mathsf{0}))\\[1em]
\textbf{function}\ \ \mathsf{pr}\,(l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\
\quad \textbf{if}\ l=\mathsf{nil}\qquad \textbf{then}\ z\\
\quad \textbf{if}\ \mathsf{car}(l)\neq\mathsf{0}\ \textbf{then}\ \mathsf{pr}(\mathsf{cdr}(l),\mathsf{car}(l)*z)\\
\qquad\qquad\qquad\ \textbf{else}\ \ \mathsf{pr}(\mathsf{cdr}(l),z)
\end{array}
$$

To transform this algorithm, we again need a technique which can handle algorithms with several recursive cases. Since $*$ is left-commutative, context moving and replacing z with $\mathsf{s}(\mathsf{0})$ results in

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{prod}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{pr}(l)\\[1em]
\textbf{function}\ \ \mathsf{pr}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\\
\quad \textbf{if}\ l=\mathsf{nil}\qquad \textbf{then}\ \mathsf{s}(\mathsf{0})\\
\quad \textbf{if}\ \mathsf{car}(l)\neq\mathsf{0}\ \textbf{then}\ \mathsf{car}(l)*\mathsf{pr}(\mathsf{cdr}(l))\\
\qquad\qquad\qquad\ \textbf{else}\ \ \mathsf{pr}(\mathsf{cdr}(l)).
\end{array}
$$

B.21 sum_digits

The following algorithm sums all digits of a natural number. It uses the algorithms div_10 and mod_10 for truncated division by 10 and for computing $x \bmod 10$.

$$\textbf{function}\ \ \mathsf{sum\_digits}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{sumd}(x, 0)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{sumd}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{sumd}(\mathsf{div\_10}(x), \mathsf{mod\_10}(x) + z) \end{array}$$

Due to the left-commutativity of $+$, context moving (and replacing z with 0) results in

$$\textbf{function}\ \ \mathsf{sum\_digits}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{sumd}(x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{sumd}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ 0 \\ \qquad\qquad \textbf{else}\ \ \mathsf{mod\_10}(x) + \mathsf{sumd}(\mathsf{div\_10}(x)). \end{array}$$

B.22 factorial

The next algorithm computes the factorial of x.

$$\textbf{function}\ \ \mathsf{factorial}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{fac}(x, \mathsf{s}(0))$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{fac}\,(x, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{fac}(\mathsf{p}(x), x * z) \end{array}$$

As $*$ is left-commutative, context moving (and replacing z with $\mathsf{s}(0)$) results in

$$\textbf{function}\ \ \mathsf{factorial}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{fac}(x)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{fac}\,(x : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ \mathsf{s}(0) \\ \qquad\qquad \textbf{else}\ \ x * \mathsf{fac}(\mathsf{p}(x)). \end{array}$$

B.23 exponent

The next algorithm computes x^y.

$$\textbf{function}\ \ \mathsf{exponent}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{exp}(x, y, \mathsf{s}(0))$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{exp}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ y = 0\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{exp}(x, \mathsf{p}(y), x * z) \end{array}$$

Similar to factorial, as $*$ is left-commutative, context moving (and replacing z with $\mathsf{s}(0)$) results in

$$\textbf{function}\ \ \mathsf{exponent}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow\ \mathsf{exp}(x, y)$$

$$\begin{array}{l} \textbf{function}\ \ \mathsf{exp}\,(x, y : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ y = 0\ \textbf{then}\ \mathsf{s}(0) \\ \qquad\qquad \textbf{else}\ \ x * \mathsf{exp}(x, \mathsf{p}(y)). \end{array}$$

B.24 exponent2

The following is again an exponentiation algorithm, but this time even numbers are treated differently from odd ones (this is similar to the algorithm multiply3 in Example B.7).

function $\mathsf{exponent2}\,(x, y : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow\; \mathsf{exp2}(x, y, \mathsf{s}(0))$

function $\mathsf{exp2}\,(x, y, z : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow$
  if $y = 0$ **then** z
  if $\mathsf{even}(y)$ **then** $\mathsf{exp2}(x * x, \mathsf{half}(y), z)$
  else $\mathsf{exp2}(x, \mathsf{p}(y), x * z)$

This algorithm requires a transformation rule that can deal with several recursive cases. Our context moving rule yields the following algorithms (where we replaced all occurrences of z by $\mathsf{s}(0)$ again).

function $\mathsf{exponent2}\,(x, y : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow\; \mathsf{exp2}(x, y)$

function $\mathsf{exp2}\,(x, y : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow$
  if $y = 0$ **then** $\mathsf{s}(0)$
  if $\mathsf{even}(y)$ **then** $\mathsf{exp2}(x * x, \mathsf{half}(y))$
  else $x * \mathsf{exp2}(x, \mathsf{p}(y))$

B.25 maximum_list

The next algorithm computes the maximum of a list. It uses the auxiliary algorithm max which returns the maximum of two numbers.

function $\mathsf{maximum\_list}\,(l : \mathsf{list}) : \mathsf{nat} \;\Leftarrow\; \mathsf{maxlist}(l, 0)$

function $\mathsf{maxlist}\,(l : \mathsf{list}, z : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow$
  if $l = \mathsf{nil}$ **then** z
  else $\mathsf{maxlist}(\mathsf{cdr}(l), \mathsf{max}(\mathsf{car}(l), z))$

As max is left-commutative, context moving (and replacing z with 0) results in

function $\mathsf{maximum\_list}\,(l : \mathsf{list}) : \mathsf{nat} \;\Leftarrow\; \mathsf{maxlist}(l)$

function $\mathsf{maxlist}\,(l) : \mathsf{nat} \;\Leftarrow$
  if $l = \mathsf{nil}$ **then** 0
  else $\mathsf{max}(\mathsf{car}(l), \mathsf{maxlist}(\mathsf{cdr}(l)))$.

Similar algorithms can also be defined on lists of list's (i.e., on objects of type llist), where the element list's can be compared by their length, for example. In this way, we also obtain left-commutative functions on list's.

B.26 minimum_list

The next algorithm computes the minimum of a list by using the auxiliary algorithm min which returns the minimum of two numbers. We assume that $\mathsf{car}(\mathsf{nil}) \equiv 0$ and $\mathsf{cdr}(\mathsf{nil}) \equiv \mathsf{nil}$.

function $\mathsf{minimum\_list}\,(l : \mathsf{list}) : \mathsf{nat} \;\Leftarrow\; \mathsf{minlist}(\mathsf{cdr}(l), \mathsf{car}(l))$

function $\mathsf{minlist}\,(l : \mathsf{list}, z : \mathsf{nat}) : \mathsf{nat} \;\Leftarrow$
  if $l = \mathsf{nil}$ **then** z
  else $\mathsf{minlist}(\mathsf{cdr}(l), \mathsf{min}(\mathsf{car}(l), z))$

As min is left-commutative, context moving results in

$$\textbf{function}\ \ \mathsf{minimum\_list}\,(l:\mathsf{list}):\mathsf{nat}\ \Leftarrow\ \mathsf{minlist}(\mathsf{cdr}(l),\mathsf{car}(l))$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{minlist}\,(l:\mathsf{list},z:\mathsf{nat}):\mathsf{nat}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{min}(\mathsf{car}(l),\mathsf{minlist}(\mathsf{cdr}(l),z)).\end{array}$$

B.27 member

The next algorithm determines whether a number x occurs in a list l. It uses an auxiliary boolean algorithm or for disjunction.

$$\textbf{function}\ \ \mathsf{member}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{bool}\ \Leftarrow\ \mathsf{mem}(x,l,\mathsf{false})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{mem}\,(x:\mathsf{nat},l:\mathsf{list},z:\mathsf{bool}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{mem}(x,\mathsf{cdr}(l),\mathsf{or}(x=\mathsf{car}(l),z))\end{array}$$

This algorithm can be transformed by context moving, as the boolean algorithm or is left-commutative. Replacing z by false afterwards yields

$$\textbf{function}\ \ \mathsf{member}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{bool}\ \Leftarrow\ \mathsf{mem}(x,l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{mem}\,(x:\mathsf{nat},l:\mathsf{list}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{false}\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{or}(x=\mathsf{car}(l),\mathsf{mem}(x,\mathsf{cdr}(l))).\end{array}$$

B.28 subset

The algorithm $\mathsf{subset}(l,k)$ returns true iff all elements of l also occur in k. It uses an auxiliary boolean algorithm and for conjunction.

$$\textbf{function}\ \ \mathsf{subset}\,(l,k:\mathsf{list}):\mathsf{bool}\ \Leftarrow\ \mathsf{sub}(l,k,\mathsf{true})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{sub}\,(l,k:\mathsf{list},z:\mathsf{bool}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{sub}(\mathsf{cdr}(l),k,\mathsf{and}(\mathsf{member}(\mathsf{car}(l),k),z))\end{array}$$

This algorithm can also be transformed by context moving, as the boolean algorithm and is left-commutative. Replacing z by true afterwards yields

$$\textbf{function}\ \ \mathsf{subset}\,(l,k:\mathsf{list}):\mathsf{bool}\ \Leftarrow\ \mathsf{sub}(l,k)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{sub}\,(l,k:\mathsf{list}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{true}\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{and}(\mathsf{member}(\mathsf{car}(l),k),\mathsf{sub}(\mathsf{cdr}(l),k)).\end{array}$$

Similarly, one may also perform analogous transformations for an algorithm which determines whether two lists are disjoint, for an algorithm which tests whether all elements in a list satisfy some property, etc.

B.29 nthcdr

The next algorithm deletes the n first elements of a list l.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{nthcdr}\,(n:\mathsf{nat}, l:\mathsf{list}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ n=0\ \textbf{then}\ l\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{nthcdr}(\mathsf{p}(n),\mathsf{cdr}(l))\end{array}$$

Here, context moving can be used to move the function cdr (of type list) to the outside.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{nthcdr}\,(n:\mathsf{nat}, l:\mathsf{list}):\mathsf{bool}\ \Leftarrow\\ \quad \textbf{if}\ n=0\ \textbf{then}\ l\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{cdr}(\mathsf{nthcdr}(\mathsf{p}(n),l))\end{array}$$

B.30 union

The next algorithm was used to introduce the context splitting rule in Section 4.

$$\textbf{function}\ \mathsf{union}\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{uni}(k,\mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \mathsf{uni}\,(k:\mathsf{llist}, z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ k\neq\mathsf{empty}\ \textbf{then}\ \mathsf{uni}(\mathsf{tl}(k),\mathsf{app}(\mathsf{hd}(k),z))\\ \qquad\qquad\qquad\quad \textbf{else}\ \ z.\end{array}$$

Here, context splitting yields

$$\textbf{function}\ \mathsf{union}\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{uni}'(k)$$

$$\begin{array}{l}\textbf{function}\ \mathsf{uni}'\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ k\neq\mathsf{empty}\ \textbf{then}\ \mathsf{app}(\mathsf{uni}'(\mathsf{tl}(k)),\mathsf{hd}(k))\\ \qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{nil}.\end{array}$$

B.31 union2

While union unites the lists in reverse order, the following alternative algorithm preserves their order.

$$\textbf{function}\ \mathsf{union2}\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{uni2}(k,\mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \mathsf{uni2}\,(k:\mathsf{llist}, z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ k\neq\mathsf{empty}\ \textbf{then}\ \mathsf{uni2}(\mathsf{tl}(k),\mathsf{app}(z,\mathsf{hd}(k)))\\ \qquad\qquad\qquad\quad \textbf{else}\ \ z.\end{array}$$

Context splitting yields

$$\textbf{function}\ \mathsf{union}\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{uni2}'(k)$$

$$\begin{array}{l}\textbf{function}\ \mathsf{uni2}'\,(k:\mathsf{llist}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ k\neq\mathsf{empty}\ \textbf{then}\ \mathsf{app}(\mathsf{hd}(k),\mathsf{uni2}'(\mathsf{tl}(k)))\\ \qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{nil}.\end{array}$$

B.32 reverse

The following list reversal algorithm was presented in Section 4 to introduce the technique of lifting constructors.

$$
\begin{array}{l}
\textbf{function} \;\; \mathsf{reverse}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \;\; \mathsf{rev}(l, \mathsf{nil}) \\[1em]
\textbf{function} \;\; \mathsf{rev}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l \neq \mathsf{nil}\ \textbf{then}\ \mathsf{rev}(\mathsf{cdr}(l), \mathsf{cons}(\mathsf{car}(l), z)) \\
\qquad\qquad\quad \textbf{else}\ \ z.
\end{array}
$$

By lifting the constructor cons to cons′, rev can be reformulated to

$$
\begin{array}{l}
\textbf{function} \;\; \mathsf{rev}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l \neq \mathsf{nil}\ \textbf{then}\ \mathsf{rev}(\mathsf{cdr}(l), \mathsf{cons}'(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), z)) \\
\qquad\qquad\quad \textbf{else}\ \ z.
\end{array}
$$

Here, cons′ computes list concatenation (i.e., it corresponds to app). Context splitting yields

$$
\begin{array}{l}
\textbf{function} \;\; \mathsf{reverse}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \;\; \mathsf{rev}'(l) \\[1em]
\textbf{function} \;\; \mathsf{rev}'\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l \neq \mathsf{nil}\ \textbf{then}\ \mathsf{cons}'(\mathsf{rev}'(\mathsf{cdr}(l)), \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})) \\
\qquad\qquad\quad \textbf{else}\ \ \mathsf{nil}.
\end{array}
$$

B.33 intersect

The next algorithm computes all those elements of l which are also contained in the list k.

$$
\begin{array}{l}
\textbf{function} \;\; \mathsf{intersect}\,(l, k : \mathsf{list}) : \mathsf{list} \Leftarrow \;\; \mathsf{int}(l, k, \mathsf{nil}) \\[1em]
\textbf{function} \;\; \mathsf{int}\,(l, k, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\qquad\quad\ \textbf{then}\ z \\
\quad \textbf{if}\ \mathsf{member}(\mathsf{car}(l), k)\ \textbf{then}\ \mathsf{int}(\mathsf{cdr}(l), k, \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil}))) \\
\qquad\qquad\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{int}(\mathsf{cdr}(l), k, z)
\end{array}
$$

Note that here we have to deal with several recursive cases. With our rule, we may perform context splitting and unfolding for app afterwards.

$$
\begin{array}{l}
\textbf{function} \;\; \mathsf{intersect}\,(l, k : \mathsf{list}) : \mathsf{list} \Leftarrow \;\; \mathsf{int}'(l, k) \\[1em]
\textbf{function} \;\; \mathsf{int}'\,(l, k : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\qquad\quad\ \textbf{then}\ \mathsf{nil} \\
\quad \textbf{if}\ \mathsf{member}(\mathsf{car}(l), k)\ \textbf{then}\ \mathsf{app}(\mathsf{int}'(\mathsf{cdr}(l), k), \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})) \\
\qquad\qquad\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{int}'(\mathsf{cdr}(l), k)
\end{array}
$$

B.34 filter

The next algorithm filters all even elements out of a list. It was introduced in Section 4 to present the technique of parameter enlargement for the auxiliary algorithm atend. Here, $\mathsf{atend}(x, y)$ inserts an element x at the end of a list y.

$$
\textbf{function} \;\; \mathsf{filter}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \;\; \mathsf{fil}(l, \mathsf{nil})
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{fil}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\quad \textbf{then}\ z \\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{fil}(\mathsf{cdr}(l), \mathsf{atend}(\mathsf{car}(l), z)) \\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{fil}(\mathsf{cdr}(l), z)
\end{array}
$$

The algorithm atend reads as follows.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atend}\,(x : \mathsf{nat}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ z = \mathsf{nil}\ \textbf{then}\ \mathsf{cons}(x, \mathsf{nil}) \\
\qquad\qquad\quad\ \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(z), \mathsf{atend}(x, \mathsf{cdr}(z)))
\end{array}
$$

Parameter enlargement of x yields

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atend}'\,(y, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\ \textbf{if}\ z = \mathsf{nil}\ \textbf{then}\ y \\
\qquad\qquad\ \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(z), \mathsf{atend}'(y, \mathsf{cdr}(z))).
\end{array}
$$

Hence, in the algorithm fil, the term $\mathsf{atend}(\mathsf{car}(l), z)$ can be replaced by $\mathsf{atend}'(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), z)$.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{fil}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\quad \textbf{then}\ z \\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{fil}(\mathsf{cdr}(l), \mathsf{atend}'(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), z)) \\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{fil}(\mathsf{cdr}(l), z)
\end{array}
$$

Now context splitting (and subsequent unfolding resp. symbolic evaluation of atend$'$) results in

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{filter}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{fil}'(l) \\
\\
\textbf{function}\ \ \mathsf{fil}'\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\quad \textbf{then}\ \mathsf{nil} \\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{cons}(\mathsf{car}(l), \mathsf{fil}'(\mathsf{cdr}(l))) \\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{fil}'(\mathsf{cdr}(l)).
\end{array}
$$

Note that here we indeed need a transformation rule which can handle algorithms with several recursive cases.

B.35 partition

The next algorithm re-orders the elements in a list such that the odd ones come before the even ones.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{partition}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{part}(l, \mathsf{nil}) \\
\\
\textbf{function}\ \ \mathsf{part}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\quad \textbf{then}\ \mathsf{app}(z, \mathsf{filter}(l)) \\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{part}(\mathsf{cdr}(l), z) \\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{part}(\mathsf{cdr}(l), \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})))
\end{array}
$$

Context splitting yields (after symbolic evaluation of app)

$$
\textbf{function}\ \ \mathsf{partition}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{part}'(l)
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{part}'\,(l:\mathsf{list}):\mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil} \qquad\quad \textbf{then}\ \mathsf{filter}(l) \\
\quad \textbf{if}\ \mathsf{even}(\mathsf{car}(l))\ \textbf{then}\ \mathsf{part}'(\mathsf{cdr}(l)) \\
\qquad\qquad\qquad\quad\ \ \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(l), \mathsf{part}'(\mathsf{cdr}(l))).
\end{array}
$$

Note that we again need a rule that can handle several recursive cases (and a non-recursive result which is not just the variable z).

Of course, an alternative definition of partition could be the following algorithm of [IB99]. Here we also need an algorithm filter_odd which works analogously to filter, but it filters out the odd elements of a list.

$$
\textbf{function}\ \ \mathsf{partition}\,(l:\mathsf{list}):\mathsf{list} \Leftarrow\ \mathsf{app}(\mathsf{filter\_odd}(l), \mathsf{filter}(l))
$$

To solve the verification problems with this algorithm, we have to apply context splitting to both filter_odd and filter (as demonstrated in Example B.34).

B.36 add_to_list

The next algorithm adds the number x to all elements in a list l. It again uses the auxiliary algorithm atend.

$$
\textbf{function}\ \ \mathsf{add\_to\_list}\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow\ \mathsf{atl}(x, l, \mathsf{nil})
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atl}\,(x:\mathsf{nat}, l, z:\mathsf{list}):\mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ z \\
\qquad\qquad\quad \textbf{else}\ \ \mathsf{atl}(x, \mathsf{cdr}(l), \mathsf{atend}(\mathsf{car}(l) + x, z))
\end{array}
$$

After replacing atend by atend′, one may perform context splitting. Subsequent unfolding resp. symbolic evaluation of atend′ yields

$$
\textbf{function}\ \ \mathsf{add\_to\_list}\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow\ \mathsf{atl}'(x, l)
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atl}'\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow \\
\quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ \mathsf{nil} \\
\qquad\qquad\quad \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(l) + x, \mathsf{atl}'(x, \mathsf{cdr}(l))).
\end{array}
$$

B.37 add_to_pos

Similar to the previous algorithm, $\mathsf{add\_to\_pos}(j, x, l)$ adds the number x to the element at position j in l. (The head of a list has position 0.)

$$
\textbf{function}\ \ \mathsf{add\_to\_pos}\,(j, x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow\ \mathsf{atp}(j, x, l, \mathsf{nil})
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atp}\,(j, x:\mathsf{nat}, l, z:\mathsf{list}):\mathsf{list} \Leftarrow \\
\quad \textbf{if}\ j = 0\ \textbf{then}\ \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l) + x, \mathsf{cdr}(l))) \\
\qquad\qquad\ \ \textbf{else}\ \ \mathsf{atp}(\mathsf{p}(j), x, \mathsf{cdr}(l), \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})))
\end{array}
$$

Context splitting and symbolic evaluation of app yields

$$
\textbf{function}\ \ \mathsf{add\_to\_pos}\,(j, x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow\ \mathsf{atp}'(j, x, l)
$$

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{atp}'\,(j, x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list} \Leftarrow \\
\quad \textbf{if}\ j = 0\ \textbf{then}\ \mathsf{cons}(\mathsf{car}(l) + x, \mathsf{cdr}(l)) \\
\qquad\qquad\ \ \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(l), \mathsf{atp}'(\mathsf{p}(j), x, \mathsf{cdr}(l))).
\end{array}
$$

B.38 insert_blanks

The following algorithm is inspired by an algorithm of the same name in [Gri81]. Given a list l and a number x, the algorithm $\mathsf{insert\_blanks}(x, l)$ adds $x * j$ to every element of l at the j-th position.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insert\_blanks}\,(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{ibl}(x, \mathsf{p}(\mathsf{length}(l)), l)\end{array}$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ibl}\,(x, j : \mathsf{nat}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ j = 0\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{ibl}(x, \mathsf{p}(j), \mathsf{add\_to\_pos}(j, x * j, z))\end{array}$$

Note that $\mathsf{add\_to\_pos}$ is left-commutative. Thus, this is an example for an algorithm where one uses a left-commutative function on list's. Context moving transforms ibl into

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ibl}\,(x, j : \mathsf{nat}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ j = 0\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{add\_to\_pos}(j, x * j, \mathsf{ibl}(x, \mathsf{p}(j), z)).\end{array}$$

B.39 insert_blanks2

The next function represents an alternative algorithm for inserting blanks.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insert\_blanks2}\,(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{ibl2}(x, 0, l, \mathsf{nil})\end{array}$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ibl2}\,(x, j : \mathsf{nat}, l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{ibl2}(x, \mathsf{s}(j), \mathsf{cdr}(l), \mathsf{atend}(x * j + \mathsf{car}(l), z))\end{array}$$

After parameter enlargement, we can perform context splitting. Subsequent symbolic evaluation yields

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insert\_blanks2}\,(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow\ \ \mathsf{ibl2}'(x, 0, l)\end{array}$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ibl2}'\,(x, j : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ z \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{cons}(x * j + \mathsf{car}(l), \mathsf{ibl2}'(x, \mathsf{s}(j), \mathsf{cdr}(l))).\end{array}$$

B.40 union3

Using the auxiliary algorithm atend, one can also formulate a version of union which works without calling the algorithm app. In this version, the order of the lists is again preserved.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{union3}\,(k : \mathsf{llist}) : \mathsf{list} \Leftarrow\ \ \mathsf{uni3}(k, \mathsf{nil})\end{array}$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{uni3}\,(k : \mathsf{llist}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ k = \mathsf{empty}\ \ \textbf{then}\ z \\ \quad \textbf{if}\ \mathsf{hd}(k) \neq \mathsf{nil}\ \textbf{then}\ \mathsf{uni3}(\mathsf{add}(\mathsf{cdr}(\mathsf{hd}(k)), \mathsf{tl}(k)), \mathsf{atend}(\mathsf{car}(\mathsf{hd}(k)), z)) \\ \qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{uni3}(\mathsf{tl}(k), z)\end{array}$$

Again by parameter enlargement, atend is replaced by atend'. Afterwards we perform context splitting (where we again need a rule which can deal with several recursive cases). Subsequent unfolding resp. symbolic evaluation of atend' yields

$$\begin{array}{l}\textbf{function}\ \ \mathsf{union3}\,(k : \mathsf{llist}) : \mathsf{list} \Leftarrow\ \ \mathsf{uni3}'(k)\end{array}$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{uni3}\,(k : \mathsf{llist}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ k = \mathsf{empty}\ \ \textbf{then}\ \mathsf{nil} \\ \quad \textbf{if}\ \mathsf{hd}(k) \neq \mathsf{nil}\ \textbf{then}\ \mathsf{cons}(\mathsf{car}(\mathsf{hd}(k)), \mathsf{uni3}'(\mathsf{add}(\mathsf{cdr}(\mathsf{hd}(k)), \mathsf{tl}(k)))) \\ \qquad\qquad\qquad\quad \textbf{else}\ \ \mathsf{uni3}'(\mathsf{tl}(k)).\end{array}$$

B.41 insert

The following algorithm inserts a natural number x into an ordered list l at the right position. It uses an algorithm $>$ to compare natural numbers.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insert}\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list}\ \Leftarrow\ \mathsf{ins}(x,l,\mathsf{nil})\\ \\ \textbf{function}\ \ \mathsf{ins}\,(x:\mathsf{nat}, l,z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\qquad \textbf{then}\ \mathsf{app}(z,\mathsf{cons}(x,\mathsf{nil}))\\ \quad \textbf{if}\ x>\mathsf{car}(l)\ \textbf{then}\ \mathsf{ins}(x,\mathsf{cdr}(l),\mathsf{app}(z,\mathsf{cons}(\mathsf{car}(l),\mathsf{nil})))\\ \qquad\qquad\qquad\ \ \textbf{else}\ \ \mathsf{app}(z,\mathsf{cons}(x,l))\end{array}$$

Here, we need our context splitting rule which can also deal with several non-recursive results. Moreover, we apply unfolding resp. symbolic evaluation to app afterwards.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insert}\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list}\ \Leftarrow\ \mathsf{ins}'(x,l)\\ \\ \textbf{function}\ \ \mathsf{ins}'\,(x:\mathsf{nat}, l:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\qquad \textbf{then}\ \mathsf{cons}(x,\mathsf{nil})\\ \quad \textbf{if}\ x>\mathsf{car}(l)\ \textbf{then}\ \mathsf{cons}(\mathsf{car}(l),\mathsf{ins}'(x,\mathsf{cdr}(l)))\\ \qquad\qquad\qquad\ \ \textbf{else}\ \ \mathsf{cons}(x,l)\end{array}$$

B.42 insertion_sort

The next algorithm sorts a list of naturals by the insertion-sort principle. It uses the algorithm insert defined above.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insertion\_sort}\,(l:\mathsf{list}):\mathsf{list}\ \Leftarrow\ \mathsf{insort}(l,\mathsf{nil})\\ \\ \textbf{function}\ \ \mathsf{insort}\,(l,z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{insort}(\mathsf{cdr}(l),\mathsf{insert}(\mathsf{car}(l),z))\end{array}$$

As insert is left-commutative, our context moving rule can be applied. Afterwards, the parameter z of insort can be replaced by the constant value nil. Thus, this is another example for a left-commutative function on list's.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{insertion\_sort}\,(l:\mathsf{list}):\mathsf{list}\ \Leftarrow\ \mathsf{insort}(l)\\ \\ \textbf{function}\ \ \mathsf{insort}\,(l:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ \mathsf{nil}\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{insert}(\mathsf{car}(l),\mathsf{insort}(\mathsf{cdr}(l)))\end{array}$$

B.43 first and second

The following functions are used to split a list into two parts. For a list l of the form $[a_0, a_1, \ldots, a_{2n}]$, $\mathsf{first}(l)$ is $[a_{2n}, \ldots, a_2, a_0]$ and $\mathsf{second}(l)$ is $[a_{2n-1}, \ldots, a_3, a_1]$. Similarly, for a list $[a_0, a_1, \ldots, a_{2n+1}]$, we obtain $\mathsf{first}(l) \equiv [a_{2n}, \ldots, a_2, a_0]$ and $\mathsf{second}(l) \equiv [a_{2n+1}, \ldots, a_3, a_1]$. We only give the algorithms for first, since the transformation of second works analogously.

$$\begin{array}{l}\textbf{function}\ \ \mathsf{first}\,(l:\mathsf{list}):\mathsf{list}\ \Leftarrow\ \mathsf{fir}(l,\mathsf{nil})\\ \\ \textbf{function}\ \ \mathsf{fir}\,(l,z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ l=\mathsf{nil}\ \textbf{then}\ z\\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{fir}(\mathsf{cdr}(\mathsf{cdr}(l)),\mathsf{cons}(\mathsf{car}(l),z))\end{array}$$

Similar to reverse, $\mathsf{cons}(\mathsf{car}(l), z))$ is lifted to $\mathsf{cons}'(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), z)$ first (where cons' computes list concatenation, i.e., it corresponds to app). Then context splitting yields

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{first}\,(l : \mathsf{list}) : \mathsf{list}\ \Leftarrow\ \mathsf{fir}'(l)\\[1em]
\textbf{function}\ \ \mathsf{fir}'\,(l : \mathsf{list}) : \mathsf{list}\ \Leftarrow\\
\quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ \mathsf{nil}\\
\qquad\qquad\quad \textbf{else}\ \ \mathsf{cons}'(\mathsf{fir}'(\mathsf{cdr}(\mathsf{cdr}(l))), \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})).
\end{array}
$$

B.44 merge

The following algorithm $\mathsf{merge}(l, k)$ merges two sorted lists l and k into one sorted list. It uses the list concatenation function app and an algorithm $\leq$ for comparing naturals.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{merge}\,(l, k : \mathsf{list}) : \mathsf{list}\ \Leftarrow\ \mathsf{mer}(l, k, \mathsf{nil})\\[1em]
\textbf{function}\ \ \mathsf{mer}\,(l, k, z : \mathsf{list}) : \mathsf{list}\ \Leftarrow\\
\quad \begin{array}{ll}
\textbf{if}\ l = \mathsf{nil} & \textbf{then}\ \mathsf{app}(z, k)\\
\textbf{if}\ k = \mathsf{nil} & \textbf{then}\ \mathsf{app}(z, l)\\
\textbf{if}\ \mathsf{car}(l) \leq \mathsf{car}(k) & \textbf{then}\ \mathsf{mer}(\mathsf{cdr}(l), k, \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})))\\
 & \textbf{else}\ \ \mathsf{mer}(l, \mathsf{cdr}(k), \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(k), \mathsf{nil})))
\end{array}
\end{array}
$$

This example demonstrates the need for a context splitting rule which can deal with algorithms that have several different recursive and several different non-recursive results. By our context splitting rule we obtain

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{merge}\,(l, k : \mathsf{list}) : \mathsf{list}\ \Leftarrow\ \mathsf{mer}'(l, k)\\[1em]
\textbf{function}\ \ \mathsf{mer}'\,(l, k : \mathsf{list}) : \mathsf{list}\ \Leftarrow\\
\quad \begin{array}{ll}
\textbf{if}\ l = \mathsf{nil} & \textbf{then}\ k\\
\textbf{if}\ k = \mathsf{nil} & \textbf{then}\ l\\
\textbf{if}\ \mathsf{car}(l) \leq \mathsf{car}(k) & \textbf{then}\ \mathsf{app}(\mathsf{cons}(\mathsf{car}(l), \mathsf{nil}), \mathsf{mer}'(\mathsf{cdr}(l), k))\\
 & \textbf{else}\ \ \mathsf{app}(\mathsf{cons}(\mathsf{car}(k), \mathsf{nil}), \mathsf{mer}'(l, \mathsf{cdr}(k))).
\end{array}
\end{array}
$$

Finally, by unfolding (resp. by symbolic evaluation) of app, the algorithm mer' can be simplified to

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{mer}'\,(l, k : \mathsf{list}) : \mathsf{list}\ \Leftarrow\\
\quad \begin{array}{ll}
\textbf{if}\ l = \mathsf{nil} & \textbf{then}\ k\\
\textbf{if}\ k = \mathsf{nil} & \textbf{then}\ l\\
\textbf{if}\ \mathsf{car}(l) \leq \mathsf{car}(k) & \textbf{then}\ \mathsf{cons}(\mathsf{car}(l), \mathsf{mer}'(\mathsf{cdr}(l), k))\\
 & \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(k), \mathsf{mer}'(l, \mathsf{cdr}(k))).
\end{array}
\end{array}
$$

B.45 merge_sort

The following function implements the "merge-sort" technique to sort lists.

$$
\begin{array}{l}
\textbf{function}\ \ \mathsf{merge\_sort}\,(l : \mathsf{list}) : \mathsf{list}\ \Leftarrow\\
\quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ \mathsf{nil}\\
\qquad\qquad\quad \textbf{else}\ \ \mathsf{merge}(\mathsf{merge\_sort}(\mathsf{first}(l)), \mathsf{merge\_sort}(\mathsf{second}(l))).
\end{array}
$$

As illustrated in Examples B.43 and B.44, the tail recursive formulations of the auxiliary algorithms first, second, and merge can be automatically transformed into a non-tail recursive form which is well suited for verification. In this way, our approach eases the verification of this implementation of merge_sort significantly.

B.46 delete

This algorithm deletes an element from a list.

function delete $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{del}(x, l, \mathsf{nil})$

function del $(x : \mathsf{nat}, l, z : \mathsf{list}) : \mathsf{list} \Leftarrow$
  if $l = \mathsf{nil}$ **then** z
  if $\mathsf{car}(l) = x$ **then** $\mathsf{app}(z, \mathsf{cdr}(l))$
  else $\mathsf{del}(x, \mathsf{cdr}(l), \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})))$

This algorithm requires the use of a transformation which can deal with several different non-recursive results. Applying our context splitting technique yields (after unfolding resp. symbolic evaluation)

function delete $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{del}'(x, l)$

function del$'$ $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow$
  if $l = \mathsf{nil}$ **then** nil
  if $\mathsf{car}(l) = x$ **then** $\mathsf{cdr}(l)$
  else $\mathsf{cons}(\mathsf{car}(l), \mathsf{del}'(x, \mathsf{cdr}(l)))$.

B.47 remove

This algorithm is similar to delete, but this time *all* occurrences of an element x are deleted from a list l.

function remove $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{rm}(x, l, \mathsf{nil})$

function rm $(x : \mathsf{nat}, l, z : \mathsf{list}) : \mathsf{list} \Leftarrow$
  if $l = \mathsf{nil}$ **then** z
  if $\mathsf{car}(l) = x$ **then** $\mathsf{rm}(x, \mathsf{cdr}(l), z)$
  else $\mathsf{rm}(x, \mathsf{cdr}(l), \mathsf{app}(z, \mathsf{cons}(\mathsf{car}(l), \mathsf{nil})))$

This algorithm requires the use of a transformation which can deal with several different recursive results. Applying our context splitting technique yields (after unfolding resp. symbolic evaluation)

function remove $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{rm}'(x, l)$

function rm$'$ $(x : \mathsf{nat}, l : \mathsf{list}) : \mathsf{list} \Leftarrow$
  if $l = \mathsf{nil}$ **then** nil
  if $\mathsf{car}(l) = x$ **then** $\mathsf{rm}'(x, \mathsf{cdr}(l))$
  else $\mathsf{cons}(\mathsf{car}(l), \mathsf{rm}'(x, \mathsf{cdr}(l)))$.

B.48 selection_sort

The next algorithm implements the selection-sort principle. It uses the auxiliary algorithm minimum_list from Example B.26 to compute the minimum of a list and it also calls the algorithm delete from Example B.46 to delete an element from a list.

function selection_sort $(l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{selsort}(l, \mathsf{nil})$

function selsort $(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow$
  if $l = \mathsf{nil}$ **then** z
  else $\mathsf{selsort}(\mathsf{delete}(\mathsf{minimum\_list}(l), l), \mathsf{atend}(\mathsf{minimum\_list}(l), z))$

We again perform parameter enlargement for atend first and replace atend(minimum_list(l), z) by atend$'$(cons(minimum_list(l), nil), z). Here, atend$'(l, k)$ computes the same result as app(k, l). Then context splitting and subsequent unfolding (resp. symbolic evaluation) yields

$$\textbf{function}\ \ \mathsf{selection\_sort}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{selsort}'(l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{selsort}'\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ \mathsf{nil} \\ \qquad\qquad \textbf{else}\ \ \mathsf{cons}(\mathsf{minimum\_list}(l), \mathsf{selsort}'(\mathsf{delete}(\mathsf{minimum\_list}(l), l))).\end{array}$$

B.49 increment

The next algorithm is used to add 1 to a number in binary representation. For that purpose, a list $l = [a_0, \ldots, a_n]$ with $a_i \in \{0, 1\}$ represents the number $\sum_{i=0,\ldots,n} a_i\, 2^i$. Thus, the first digit of the list is the least significant bit. Here, we assume that car(nil) $\equiv$ 0.

$$\textbf{function}\ \ \mathsf{increment}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{inc}(l, \mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{inc}\,(l, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ \mathsf{car}(l) = 0\ \textbf{then}\ \mathsf{app}(z, \mathsf{cons}(1, \mathsf{cdr}(l))) \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{inc}(\mathsf{cdr}(l), \mathsf{app}(z, \mathsf{cons}(0, \mathsf{nil})))\end{array}$$

Context splitting and subsequent unfolding (resp. symbolic evaluation) yields

$$\textbf{function}\ \ \mathsf{increment}\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \mathsf{inc}'(l)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{inc}'\,(l : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ \mathsf{car}(l) = 0\ \textbf{then}\ \mathsf{cons}(1, \mathsf{cdr}(l)) \\ \qquad\qquad\quad \textbf{else}\ \ \mathsf{cons}(0, \mathsf{inc}'(\mathsf{cdr}(l))).\end{array}$$

B.50 base

This algorithm converts a natural number x into its representation w.r.t. base y (where this time the leftmost digit of the resulting number should be the most significant one). For that purpose it uses the algorithm quotient from Example B.13 (where quot(x, y) computes $\lfloor \frac{x}{y} \rfloor$) and an algorithm mod.

$$\textbf{function}\ \ \mathsf{base}\,(x, y : \mathsf{nat}) : \mathsf{list} \Leftarrow \mathsf{ba}(x, y, \mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ba}\,(x, y : \mathsf{nat}, z : \mathsf{list}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{ba}(\mathsf{quotient}(x, y), y, \mathsf{cons}(\mathsf{mod}(x, y), z))\end{array}$$

After lifting cons to cons$'$ (resp. to app), we can apply context splitting. Subsequent unfolding (resp. symbolic evaluation) yields

$$\textbf{function}\ \ \mathsf{base}\,(x, y : \mathsf{nat}) : \mathsf{list} \Leftarrow \mathsf{ba}'(x, y)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{ba}'\,(x, y : \mathsf{nat}) : \mathsf{list} \Leftarrow \\ \quad \textbf{if}\ x = 0\ \textbf{then}\ \mathsf{nil} \\ \qquad\qquad \textbf{else}\ \ \mathsf{app}(\mathsf{ba}'(\mathsf{quotient}(x, y), y), \mathsf{cons}(\mathsf{mod}(x, y), \mathsf{nil}).\end{array}$$

B.51 column

List of lists B (i.e., objects of type llist) can be used to model matrices. For that purpose, every list in B represents a row in the matrix. The following algorithm returns the first column of a matrix B.

$$\textbf{function}\ \ \mathsf{column}\,(B : \mathsf{llist}) : \mathsf{list} \ \Leftarrow\ \mathsf{col}(B, \mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{col}\,(B : \mathsf{llist}, z : \mathsf{list}) : \mathsf{list} \ \Leftarrow \\ \quad \textbf{if}\ B = \mathsf{empty}\ \textbf{then}\ z \\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{col}(\mathsf{tl}(B), \mathsf{atend}(z, \mathsf{car}(\mathsf{hd}(B))))\end{array}$$

After parameter enlargement for atend we can apply context splitting. Subsequent unfolding (resp. symbolic evaluation) yields

$$\textbf{function}\ \ \mathsf{column}\,(B : \mathsf{llist}) : \mathsf{list} \ \Leftarrow\ \mathsf{col}'(B)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{col}'\,(B : \mathsf{llist}) : \mathsf{list} \ \Leftarrow \\ \quad \textbf{if}\ B = \mathsf{empty}\ \textbf{then}\ \mathsf{nil} \\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{cons}(\mathsf{car}(\mathsf{hd}(B)), \mathsf{col}'(\mathsf{tl}(B)))\end{array}$$

B.52 but_column

Similar to the preceding algorithm, this algorithm deletes the first column from a matrix B. It uses an algorithm $\mathsf{atend\_ll} : \mathsf{list} \times \mathsf{llist} \to \mathsf{llist}$ which works analogously to atend, but it adds a list to the end of a list of list's.

$$\textbf{function}\ \ \mathsf{but\_column}\,(B : \mathsf{llist}) : \mathsf{llist} \ \Leftarrow\ \mathsf{butcol}(B, \mathsf{empty})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{butcol}\,(B : \mathsf{llist}, z : \mathsf{llist}) : \mathsf{llist} \ \Leftarrow \\ \quad \textbf{if}\ B = \mathsf{empty}\ \textbf{then}\ z \\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{butcol}(\mathsf{tl}(B), \mathsf{atend\_ll}(z, \mathsf{cdr}(\mathsf{hd}(B))))\end{array}$$

After parameter enlargement for atend_ll we can apply context splitting. Subsequent unfolding (resp. symbolic evaluation) yields

$$\textbf{function}\ \ \mathsf{but\_column}\,(B : \mathsf{llist}) : \mathsf{llist} \ \Leftarrow\ \mathsf{butcol}'(B)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{butcol}'\,(B : \mathsf{llist}) : \mathsf{llist} \ \Leftarrow \\ \quad \textbf{if}\ B = \mathsf{empty}\ \textbf{then}\ \mathsf{empty} \\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{add}(\mathsf{cdr}(\mathsf{hd}(B)), \mathsf{butcol}'(\mathsf{tl}(B)))\end{array}$$

B.53 scalar_product

The next algorithm computes the scalar product of two vectors l and k (modelled by list's).

$$\textbf{function}\ \ \mathsf{scalar\_product}\,(l, k : \mathsf{list}) : \mathsf{nat} \ \Leftarrow\ \mathsf{scalar}(l, k, 0)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{scalar}\,(l, k : \mathsf{list}, z : \mathsf{nat}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ z \\ \qquad\qquad \textbf{else}\ \ \mathsf{scalar}(\mathsf{cdr}(l), \mathsf{cdr}(k), \mathsf{car}(l) * \mathsf{car}(k) + z)\end{array}$$

As $+$ is left-commutative we can apply context moving. Subsequent replacement of the variable z by 0 yields

$$\textbf{function}\ \ \mathsf{scalar\_product}\,(l, k : \mathsf{list}) : \mathsf{nat} \ \Leftarrow\ \mathsf{scalar}(l, k)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{scalar}\,(l, k : \mathsf{list}) : \mathsf{nat} \ \Leftarrow \\ \quad \textbf{if}\ l = \mathsf{nil}\ \textbf{then}\ 0 \\ \qquad\qquad \textbf{else}\ \ \mathsf{car}(l) * \mathsf{car}(k) + \mathsf{scalar}(\mathsf{cdr}(l), \mathsf{cdr}(k)).\end{array}$$

B.54 vec_matrix

The algorithm vec_matrix computes the multiplication of a vector a with a matrix B using the auxiliary algorithms defined above.

$$\textbf{function}\ \ \mathsf{vec\_matrix}\,(a:\mathsf{list}, B:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{vm}(a,B,\mathsf{nil})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{vm}\,(a:\mathsf{list}, B:\mathsf{llist}, z:\mathsf{list}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ B=\mathsf{empty}\ \textbf{then}\ z\\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{vm}(a,\mathsf{but\_column}(B),\mathsf{atend}(z,\mathsf{scalar\_product}(a,\mathsf{column}(B))))\end{array}$$

By parameter enlargement (for atend), context splitting, and symbolic evaluation we obtain

$$\textbf{function}\ \ \mathsf{vec\_matrix}\,(a:\mathsf{list}, B:\mathsf{llist}):\mathsf{list}\ \Leftarrow\ \mathsf{vm}'(a,B)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{vm}'\,(a:\mathsf{list}, B:\mathsf{llist}):\mathsf{list}\ \Leftarrow\\ \quad \textbf{if}\ B=\mathsf{empty}\ \textbf{then}\ \mathsf{nil}\\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{cons}(\mathsf{scalar\_product}(a,\mathsf{column}(B)),\mathsf{vm}'(a,\mathsf{but\_column}(B)))\end{array}$$

B.55 matrix_mult

Similar to the previous algorithm, the following algorithm computes matrix multiplication.

$$\textbf{function}\ \ \mathsf{matrix\_mult}\,(A,B:\mathsf{llist}):\mathsf{llist}\ \Leftarrow\ \mathsf{mm}(A,B,\mathsf{empty})$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{mm}\,(A,B,z:\mathsf{llist}):\mathsf{llist}\ \Leftarrow\\ \quad \textbf{if}\ A=\mathsf{empty}\ \textbf{then}\ z\\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{mm}(\mathsf{tl}(A),B,\mathsf{atend\_ll}(z,\mathsf{vec\_matrix}(\mathsf{hd}(A),B)))\end{array}$$

By parameter enlargement (for atend_ll), context splitting, and symbolic evaluation we obtain

$$\textbf{function}\ \ \mathsf{matrix\_mult}\,(A,B:\mathsf{llist}):\mathsf{llist}\ \Leftarrow\ \mathsf{mm}'(A,B)$$

$$\begin{array}{l}\textbf{function}\ \ \mathsf{mm}\,(A,B:\mathsf{llist}):\mathsf{llist}\ \Leftarrow\\ \quad \textbf{if}\ A=\mathsf{empty}\ \textbf{then}\ \mathsf{empty}\\ \qquad\qquad\qquad \textbf{else}\ \ \mathsf{add}(\mathsf{vec\_matrix}(\mathsf{hd}(A),B),\mathsf{mm}'(\mathsf{tl}(A),B)).\end{array}$$

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