

# Termination of Polynomial Loops<sup>\*</sup>

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**Abstract.** We consider the termination problem for triangular weakly non-linear loops (*tw*n-loops) over some ring  $\mathcal{S}$  like  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{R}$ . Essentially, the guard of such a loop is an arbitrary Boolean formula over (possibly non-linear) polynomial inequations, and the body is a single assignment

$\begin{bmatrix} x_1 \\ \dots \\ x_d \end{bmatrix} \leftarrow \begin{bmatrix} c_1 \cdot x_1 + p_1 \\ \dots \\ c_d \cdot x_d + p_d \end{bmatrix}$  where each  $x_i$  is a variable,  $c_i \in \mathcal{S}$ , and each  $p_i$  is a

(possibly non-linear) polynomial over  $\mathcal{S}$  and the variables  $x_{i+1}, \dots, x_d$ . We present a reduction from the question of termination to the existential fragment of the first-order theory of  $\mathcal{S}$  and  $\mathbb{R}$ . For loops over  $\mathbb{R}$ , our reduction entails decidability of termination. For loops over  $\mathbb{Z}$  and  $\mathbb{Q}$ , it proves semi-decidability of non-termination.

Furthermore, we present a transformation to convert certain non-*tw*n-loops into *tw*n-form. Then the original loop terminates iff the transformed loop terminates over a specific subset of  $\mathbb{R}$ , which can also be checked via our reduction. This transformation also allows us to prove *tight* complexity bounds for the termination problem for two important classes of loops which can *always* be transformed into *tw*n-loops.

## 1 Introduction

Let  $\mathbb{R}_{\mathbb{A}}$  denote the real algebraic numbers. We consider loops of the form

$$\mathbf{while} \ \varphi \ \mathbf{do} \ \vec{x} \leftarrow \vec{a}. \tag{1}$$

Here,  $\vec{x}$  is a vector<sup>3</sup> of  $d \geq 1$  pairwise different variables that range over a ring  $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}_{\mathbb{A}}$ , where  $\leq$  denotes the subring relation. Moreover,  $\vec{a} \in (\mathcal{S}[\vec{x}])^d$  where  $\mathcal{S}[\vec{x}]$  is the set of all polynomials over  $\vec{x}$  with coefficients from  $\mathcal{S}$ . The condition  $\varphi$  is an arbitrary propositional formula over the atoms  $\{p \triangleright 0 \mid p \in \mathcal{S}[\vec{x}], \triangleright \in \{\geq, >\}\}$ .<sup>4</sup>

We require  $\mathcal{S} \leq \mathbb{R}_{\mathbb{A}}$  instead of  $\mathcal{S} \leq \mathbb{R}$ , as it is unclear how to represent transcendental numbers on computers. However, in [Sect. 5](#) we will see that the

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<sup>3</sup> We use row- and column-vectors interchangeably to improve readability.

<sup>4</sup> Note that negation is syntactic sugar in our setting, as, e.g.,  $-(p > 0)$  is equivalent to  $-p \geq 0$ . So w.l.o.g.  $\varphi$  is built from atoms,  $\wedge$ , and  $\vee$ .

loops considered in this paper terminate over  $\mathbb{R}$  iff they terminate over  $\mathbb{R}_A$ . Thus, our results immediately carry over to loops where the variables range over  $\mathbb{R}$ . Hence, we sometimes also consider loops over  $\mathcal{S} = \mathbb{R}$ . However, even then we restrict ourselves to loops (1) where all constants in  $\varphi$  and  $\vec{a}$  are algebraic.

We often represent a loop (1) by the tuple  $(\varphi, \vec{a})$  of the *loop condition*  $\varphi$  and the *update*  $\vec{a} = (a_1, \dots, a_d)$ . Unless stated otherwise,  $(\varphi, \vec{a})$  is always a loop on  $\mathcal{S}^d$  using the variables  $\vec{x} = (x_1, \dots, x_d)$  where  $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}_A$ . A *linear-update loop* has the form  $(\varphi, A \cdot \vec{x} + \vec{b})$  and it has *real spectrum* if  $A$  has real eigenvalues only. A *linear loop* is a linear-update loop where  $\varphi$  is linear (i.e., its atoms are only linear<sup>5</sup> inequations). A *conjunctive loop* is a loop  $(\varphi, \vec{a})$  where  $\varphi$  does not contain disjunctions.

There exist several decidability results for the termination of linear loops [6, 8, 15, 24, 34, 37, 41, 53], but there are only very few results on the decidability of termination for certain forms of non-linear loops [35, 36, 38, 55]. Moreover, all of these previous works only consider conjunctive loops besides [38] which only allows for loop conditions defining compact sets. In this paper, we regard (linear and non-linear) loops with arbitrary conditions, i.e., they may also contain disjunctions and define non-compact sets. Furthermore, we study the decidability of termination for non-linear loops over  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}_A$ , and  $\mathbb{R}$ , whereas the existing decidability results for non-linear loops are restricted to loops over  $\mathbb{R}$ . So we identify new sub-classes of loops of the form (1) where (non-)termination is (semi-)decidable. Moreover, we also investigate the complexity of the termination problem.

*Contributions:* We study a sub-class of loops of the form (1) (so-called *twn-loops* (Sect. 2)), and present an (incomplete) transformation  $Tr$  from non-*twn-loops* to *twn-loops* (Sect. 3). Then we show that termination of *twn-loops* over  $\mathbb{R}_A$  and  $\mathbb{R}$  is decidable and that non-termination over  $\mathbb{Z}$  and  $\mathbb{Q}$  is semi-decidable (Sect. 4 and 5). For those classes of non-*twn-loops* where our transformation  $Tr$  is complete, we obtain analogous decidability results. For all other loops of the form (1), our (semi-)decision procedures still apply if  $Tr$  is applicable.

Finally, we prove Co-NP-completeness of termination of linear loops over  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}_A$ , and  $\mathbb{R}$  with real spectrum and  $\forall\mathbb{R}$ -completeness of termination of linear-update loops with real spectrum over  $\mathbb{R}_A$  and  $\mathbb{R}$  (Sect. 6).

All missing proofs can be found in [16].

## 2 Preliminaries

For any entity  $s$ ,  $s[x/t]$  is the entity that results from  $s$  by replacing all free occurrences of  $x$  by  $t$ . Similarly, if  $\vec{x} = (x_1, \dots, x_d)$  and  $\vec{t} = (t_1, \dots, t_d)$ , then  $s[\vec{x}/\vec{t}]$  results from  $s$  by replacing all free occurrences of  $x_i$  by  $t_i$ , for each  $1 \leq i \leq d$ .

Any vector of polynomials  $\vec{a} \in (\mathcal{S}[\vec{x}])^d$  can also be regarded as a function  $\vec{a} : (\mathcal{S}[\vec{x}])^d \rightarrow (\mathcal{S}[\vec{x}])^d$ , where for any  $\vec{p} \in (\mathcal{S}[\vec{x}])^d$ ,  $\vec{a}(\vec{p}) = \vec{a}[\vec{x}/\vec{p}]$  results from

<sup>5</sup> In this paper “linear” refers to “linear polynomials” and thus includes *affine* functions.

applying the polynomials  $\vec{a}$  to the polynomials  $\vec{p}$ . In a similar way, we can also apply a formula to polynomials  $\vec{p} \in (\mathcal{S}[\vec{x}])^d$ . To this end, we define  $\psi(\vec{p}) = \psi[\vec{x}/\vec{p}]$  for first-order formulas  $\psi$  with free variables  $\vec{x}$ . As usual, function application associates to the left, i.e.,  $\vec{a}(\vec{b})(\vec{p})$  stands for  $(\vec{a}(\vec{b}))(\vec{p})$ . However, since applying polynomials only means that one instantiates variables, we obviously have  $(\vec{a}(\vec{b}))(\vec{p}) = \vec{a}(\vec{b}(\vec{p}))$ .

Def. 1 formalizes the intuitive notion of termination for a loop  $(\varphi, \vec{a})$ .

**Definition 1 (Termination).** *The loop  $(\varphi, \vec{a})$  is non-terminating (over  $\mathcal{S}$ ) if*

$$\exists \vec{c} \in \mathcal{S}^d. \forall n \in \mathbb{N}. \varphi(\vec{a}^n(\vec{c})).$$

*Then  $\vec{c}$  is a witness for non-termination. Otherwise,  $(\varphi, \vec{a})$  terminates (over  $\mathcal{S}$ ).*

Here,  $\vec{a}^n$  denotes the  $n$ -fold application of  $\vec{a}$ , i.e.,  $\vec{a}^0(\vec{c}) = \vec{c}$  and  $\vec{a}^{n+1}(\vec{c}) = \vec{a}(\vec{a}^n(\vec{c}))$ . Termination (which is sometimes also called *universal* termination) is not to be confused with the *halting problem*, where one is interested in termination w.r.t. a *given* input. In contrast, Def. 1 considers termination w.r.t. *all* inputs.

For any entity  $s$ , let  $\mathcal{V}(s)$  be the set of all free variables that occur in  $s$ . Given an assignment  $\vec{x} \leftarrow \vec{a}$ , the relation  $\succ_{\vec{a}} \in \mathcal{V}(\vec{a}) \times \mathcal{V}(\vec{a})$  is the transitive closure of  $\{(x_i, x_j) \mid i, j \in \{1, \dots, d\}, i \neq j, x_j \in \mathcal{V}(a_i)\}$ . We call  $(\varphi, \vec{a})$  *triangular* if  $\succ_{\vec{a}}$  is well founded. So the restriction to triangular loops prohibits “cyclic dependencies” of variables (e.g., where the new values of  $x_1$  and  $x_2$  both depend on the old values of  $x_1$  and  $x_2$ ). For example, a loop with the body  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \\ x_2 - 1^2 \end{bmatrix}$  is triangular since  $\succ = \{(x_1, x_2)\}$  is well founded, whereas a loop with the body  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \\ x_1 - 1^2 \end{bmatrix}$  is not triangular. Triangularity is used to compute a *closed form* for the  $n$ -fold application of the loop update  $\vec{a}$ , i.e., a vector  $\vec{q}$  of  $d$  expressions over the variables  $\vec{x}$  and  $n$  such that  $\vec{q} = \vec{a}^n$ . From a practical point of view, the restriction to triangular loops seems quite natural. For example, in [18], 1511 polynomial loops were extracted from the *Termination Problems Data Base* [54], the benchmark collection which is used at the annual *Termination and Complexity Competition* [21], and only 26 of them were non-triangular.

The loop  $(\varphi, \vec{a})$  is *weakly non-linear* if for no  $i$ ,  $x_i$  occurs in a non-linear monomial of  $a_i$ . So for example, a loop with the body  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \\ x_2 - 1^2 \end{bmatrix}$  is weakly non-linear, whereas a loop with the body  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \cdot x_2 \\ x_2 - 1 \end{bmatrix}$  is not. Together with triangularity, weak non-linearity ensures that we can always compute closed forms. In particular, weak non-linearity excludes loops like  $(\varphi, x \leftarrow x^2)$  that need exponential space, as the values of some variables grow doubly exponentially.

A *twn*-loop is triangular and weakly non-linear. So in other words, by permuting variables every *twn*-loop can be transformed to the form

$$\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \leftarrow \begin{bmatrix} c_1 \cdot x_1 + p_1 \\ \vdots \\ c_d \cdot x_d + p_d \end{bmatrix}$$

where  $c_i \in \mathcal{S}$  and  $p_i \in \mathcal{S}[x_{i+1}, \dots, x_d]$ . If  $(\varphi, \vec{a})$  is weakly non-linear and  $x_i$ 's coefficient in  $a_i$  is non-negative for all  $1 \leq i \leq d$ , then  $(\varphi, \vec{a})$  is *non-negative*. A *tnn*-loop is triangular and non-negative (and thus, also weakly non-linear).

Our *twn*-loops are a special case of *solvable loops* [48].

**Definition 2 (Solvable Loops).** A loop  $(\varphi, \vec{a})$  is solvable if there is a partitioning  $\mathcal{J} = \{J_1, \dots, J_k\}$  of  $\{1, \dots, d\}$  such that for each  $1 \leq i \leq k$  we have

$$\vec{a}_{J_i} = A_i \cdot \vec{x}_{J_i} + \vec{p}_i,$$

where  $\vec{a}_{J_i}$  is the vector of all  $a_j$  with  $j \in J_i$  (and  $\vec{x}_{J_i}$  is defined analogously),  $d_i = |J_i|$ ,  $A_i \in \mathcal{S}^{d_i \times d_i}$ , and  $\vec{p}_i \in (\mathcal{S}[\vec{x}_{J_{i+1}}, \dots, \vec{x}_{J_k}])^{d_i}$ . The eigenvalues of a solvable loop are defined as the union of the eigenvalues of all  $A_i$ .

So solvable loops allow for blocks of variables with linear dependencies, and *twn*-loops correspond to the case that each such block has size 1. While our approach could easily be generalized to solvable loops with real eigenvalues, in [Sect. 3](#) we show that such a generalization does not increase its applicability.

For a ring  $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}_{\mathbb{A}}$ , the *existential fragment of the first-order theory of  $\mathcal{S}$*  is the set  $\text{Th}_{\exists}(\mathcal{S})$  of all formulas  $\exists \vec{y} \in \mathcal{S}^k. \psi$ , where  $\psi$  is a propositional formula over the atoms  $\{p \triangleright 0 \mid p \in \mathbb{Q}[\vec{y}, \vec{z}], \triangleright \in \{\geq, >\}\}$  and  $k \in \mathbb{N}$  [[44](#), [51](#)]. Here,  $\vec{y}$  and  $\vec{z}$  are pairwise disjoint vectors of variables (i.e., the variables  $\vec{z}$  are *free*). Moreover,  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  is the set of all formulas  $\exists \vec{y}' \in \mathbb{R}_{\mathbb{A}}^{k'}, \vec{y} \in \mathcal{S}^k. \psi$ , with a propositional formula  $\psi$  over  $\{p \triangleright 0 \mid p \in \mathbb{Q}[\vec{y}', \vec{y}, \vec{z}], \triangleright \in \{\geq, >\}\}$  where  $k', k \in \mathbb{N}$  and the variables  $\vec{y}'$ ,  $\vec{y}$ , and  $\vec{z}$  are pairwise disjoint. As usual, a formula without free variables is *closed*. In the following, we also consider formulas over inequations  $p \triangleright 0$  where  $p$ 's coefficients are from  $\mathbb{R}_{\mathbb{A}}$  to be elements of  $\text{Th}_{\exists}(\mathbb{R}_{\mathbb{A}})$  (resp.  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ ). The reason is that real algebraic numbers are  $\text{Th}_{\exists}(\mathbb{R}_{\mathbb{A}})$ -definable.

Finally, note that validity of formulas from  $\text{Th}_{\exists}(\mathcal{S})$  or  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  is decidable if  $\mathcal{S} \in \{\mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$  and semi-decidable if  $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}\}$  [[11](#), [52](#)]. By undecidability of Hilbert's Tenth Problem, validity is undecidable for  $\mathcal{S} = \mathbb{Z}$ . While validity of full first-order formulas (i.e., also containing universal quantifiers) over  $\mathcal{S} = \mathbb{Q}$  is undecidable [[45](#)], it is still open whether validity of formulas from  $\text{Th}_{\exists}(\mathbb{Q})$  or  $\text{Th}_{\exists}(\mathbb{Q}, \mathbb{R}_{\mathbb{A}})$  is decidable. However, validity of *linear* formulas from  $\text{Th}_{\exists}(\mathcal{S})$  or  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  is decidable for all  $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$ .

### 3 Transformation to Triangular Weakly Non-Linear Form

We first show how to handle loops that are not yet *twn*. To this end, we introduce a transformation of loops via *polynomial automorphisms* in [Sect. 3.1](#) and show that our transformation preserves (non-)termination ([Thm. 10](#)). In [Sect. 3.2](#), we use results from algebraic geometry to show that the question whether a loop can be transformed into *twn*-form is reducible to validity of  $\text{Th}_{\exists}(\mathbb{R}_{\mathbb{A}})$ -formulas ([Thm. 20](#)). Moreover, we show that it is decidable whether a *linear* automorphism can transform a loop into a special case of the *twn*-form ([Thm. 23](#)).

#### 3.1 Transforming Loops

Clearly, the *polynomials*  $x_1, \dots, x_d$  are *generators* of the  $\mathcal{S}$ -algebra  $\mathcal{S}[\vec{x}]$ , i.e., every polynomial from  $\mathcal{S}[\vec{x}]$  can be obtained from  $x_1, \dots, x_d$  and the operations of the algebra (i.e., addition and multiplication). So far, we have implicitly chosen a special “representation” of the loop based on the generators  $x_1, \dots, x_d$ .

We now change this representation, i.e., we use a different set of  $d$  polynomials which are also generators of  $\mathcal{S}[\vec{x}]$ . Then the loop has to be modified accordingly in order to adapt it to this new representation. This modification does not affect the loop's termination behavior, but it may transform a non-*twn*-loop into *twn*-form.

The desired change of representation is described by  $\mathcal{S}$ -automorphisms of  $\mathcal{S}[\vec{x}]$ . As usual, an  $\mathcal{S}$ -endomorphism of  $\mathcal{S}[\vec{x}]$  is a mapping  $\eta : \mathcal{S}[\vec{x}] \rightarrow \mathcal{S}[\vec{x}]$  which is  $\mathcal{S}$ -linear and multiplicative.<sup>6</sup> We denote the ring of  $\mathcal{S}$ -endomorphisms of  $\mathcal{S}[\vec{x}]$  by  $\text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  (where the operations on this ring are pointwise addition and function composition  $\circ$ ). The group of  $\mathcal{S}$ -automorphisms of  $\mathcal{S}[\vec{x}]$  is  $\text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ 's group of units, and we denote it by  $\text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ . So an  $\mathcal{S}$ -automorphism of  $\mathcal{S}[\vec{x}]$  is an  $\eta \in \text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  that is *invertible*. Thus, there exists an  $\eta^{-1} \in \text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  such that  $\eta \circ \eta^{-1} = \eta^{-1} \circ \eta = \text{id}_{\mathcal{S}[\vec{x}]}$ , where  $\text{id}_{\mathcal{S}[\vec{x}]}$  is the identity function on  $\mathcal{S}[\vec{x}]$ .

*Example 3 (Automorphism).* Let  $\eta \in \text{End}_{\mathcal{S}}(\mathcal{S}[x_1, x_2])$  with  $\eta(x_1) = x_2$ ,  $\eta(x_2) = x_1 - x_2^2$ . Then  $\eta \in \text{Aut}_{\mathcal{S}}(\mathcal{S}[x_1, x_2])$ , where  $\eta^{-1}(x_1) = x_1^2 + x_2$  and  $\eta^{-1}(x_2) = x_1$ .

As  $\mathcal{S}[\vec{x}]$  is free on the generators  $\vec{x}$ , an endomorphism  $\eta \in \text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  is uniquely determined by the images of the variables, i.e., by  $\eta(x_1), \dots, \eta(x_d)$ . Hence, we have a one-to-one correspondence between elements of  $(\mathcal{S}[\vec{x}])^d$  and  $\text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ . In particular, every tuple  $\vec{a} = (a_1, \dots, a_d) \in (\mathcal{S}[\vec{x}])^d$  corresponds to the unique endomorphism  $\tilde{a} \in \text{End}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  with  $\tilde{a}(x_i) = a_i$  for all  $1 \leq i \leq d$ . So for any  $p \in \mathcal{S}[\vec{x}]$  we have  $\tilde{a}(p) = p(\vec{a})$ . Thus, the update of a loop induces an endomorphism which operates on polynomials.

*Example 4 (Updates as Endomorphisms).* Consider the loop

**while**  $x_2^3 + x_1 - x_2^2 > 0$  **do**  $(x_1, x_2) \leftarrow (a_1, a_2)$

where  $a_1 = ((-x_2^2 + x_1)^2 + x_2)^2 - 2 \cdot x_2^2 + 2 \cdot x_1$  and  $a_2 = (-x_2^2 + x_1)^2 + x_2$ , i.e.,  $\varphi = (x_2^3 + x_1 - x_2^2 > 0)$  and  $\vec{a} = (a_1, a_2)$ . Then  $\vec{a}$  induces the endomorphism  $\tilde{a}$  with  $\tilde{a}(x_1) = a_1$  and  $\tilde{a}(x_2) = a_2$ . So we have  $\tilde{a}(2 \cdot x_1 + x_2^3) = (2 \cdot x_1 + x_2^3)(\vec{a}) = 2 \cdot a_1 + a_2^3$ .

For tuples of numbers (e.g.,  $\vec{c} = (5, 2)$ ), the endomorphism  $\tilde{c}$  is  $\tilde{c}(x_1) = 5$  and  $\tilde{c}(x_2) = 2$ . Thus, we have  $\tilde{c}(x_2^3 + x_1 - x_2^2) = (x_2^3 + x_1 - x_2^2)(\vec{c}) = 2^3 + 5 - 2^2 = 9$ .

We extend the application of endomorphisms  $\eta : \mathcal{S}[\vec{x}] \rightarrow \mathcal{S}[\vec{x}]$  to vectors of polynomials  $\vec{a} = (a_1, \dots, a_d)$  by defining  $\eta(\vec{a}) = (\eta(a_1), \dots, \eta(a_d))$  and to formulas  $\varphi \in \text{Th}_{\exists}(\mathcal{S})$  by defining  $\eta(\varphi) = \varphi(\eta(\vec{x}))$ , i.e.,  $\eta(\varphi)$  results from  $\varphi$  by applying  $\eta$  to all polynomials that occur in  $\varphi$ . This allows us to transform  $(\varphi, \vec{a})$  into a new loop  $\text{Tr}_{\eta}(\varphi, \vec{a})$  using any automorphism  $\eta \in \text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ .

**Definition 5 (Tr).** Let  $\eta \in \text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ . We define  $\text{Tr}_{\eta}(\varphi, \vec{a}) = (\varphi', \vec{a}')$  where

$$\varphi' = \eta^{-1}(\varphi) \quad \text{and} \quad \vec{a}' = (\eta^{-1} \circ \tilde{a} \circ \eta)(\vec{x}).^7$$

<sup>6</sup> So we have  $\eta(c \cdot p + c' \cdot p') = c \cdot \eta(p) + c' \cdot \eta(p')$ ,  $\eta(1) = 1$ , and  $\eta(p \cdot p') = \eta(p) \cdot \eta(p')$  for all  $c, c' \in \mathcal{S}$  and all  $p, p' \in \mathcal{S}[\vec{x}]$ .

<sup>7</sup> In other words, we have  $\vec{a}' = (\eta(\vec{x}))(\vec{a})(\eta^{-1}(\vec{x}))$ , since  $(\eta^{-1} \circ \tilde{a} \circ \eta)(\vec{x}) = \eta^{-1}(\eta(\vec{x})[\vec{x}/\vec{a}]) = \eta(\vec{x})[\vec{x}/\vec{a}][\vec{x}/\eta^{-1}(\vec{x})] = (\eta(\vec{x}))(\vec{a})(\eta^{-1}(\vec{x}))$ .

*Example 6 (Transforming Loops).* We transform the loop  $(\varphi, \vec{a})$  from [Ex. 4](#) with the automorphism  $\eta$  from [Ex. 3](#). We obtain  $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$  where

$$\begin{aligned}\varphi' &= \eta^{-1}(\varphi) = ((\eta^{-1}(x_2))^3 + \eta^{-1}(x_1) - (\eta^{-1}(x_2))^2 > 0) \\ &= (x_1^3 + x_1^2 + x_2 - x_1^2 > 0) = (x_1^3 + x_2 > 0) \quad \text{and} \\ \vec{a}' &= ((\eta^{-1} \circ \tilde{a} \circ \eta)(x_1), (\eta^{-1} \circ \tilde{a} \circ \eta)(x_2)) = (\eta^{-1}(\tilde{a}(x_2)), \eta^{-1}(\tilde{a}(x_1 - x_2^2))) \\ &= (\eta^{-1}(a_2), \eta^{-1}(a_1 - a_2^2)) = (x_1 + x_2^2, 2 \cdot x_2).\end{aligned}$$

So the resulting transformed loop is  $(x_1^3 + x_2 > 0, (x_1 + x_2^2, 2 \cdot x_2))$ . Note that while the original loop  $(\varphi, \vec{a})$  is neither triangular nor weakly non-linear, the resulting transformed loop is *tw*n. Also note that we used a *non-linear* automorphism with  $\eta(x_2) = x_1 - x_2^2$  for the transformation.

While the above example shows that our transformation can indeed transform non-*tw*n-loops into *tw*n-loops, it remains to prove that this transformation preserves (non-)termination. Then we can use our techniques for termination analysis of *tw*n-loops for *tw*n-transformable-loops as well, i.e., for all loops  $(\varphi, \vec{a})$  where  $Tr_\eta(\varphi, \vec{a})$  is *tw*n for some automorphism  $\eta$ . (The question how to find such automorphisms will be addressed in [Sect. 3.2](#).)

As a first step, by [Lemma 7](#), our transformation is “compatible” with the operation  $\circ$  of the group  $\text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ , i.e., it is an *action*.

**Lemma 7.** *Tr is an action of  $\text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$  on loops, i.e., for  $\eta_1, \eta_2 \in \text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$*

$$Tr_{id_{\mathcal{S}[\vec{x}]}}(\varphi, \vec{a}) = (\varphi, \vec{a}) \quad \text{and} \quad Tr_{\eta_1 \circ \eta_2}(\varphi, \vec{a}) = Tr_{\eta_2}(Tr_{\eta_1}(\varphi, \vec{a})).$$

The next lemma shows that a witness for non-termination of  $(\varphi, \vec{a})$  is transformed by  $\eta(\vec{x})$  into a witness for non-termination of  $Tr_\eta(\varphi, \vec{a})$ .

**Lemma 8.** *If  $\vec{c}$  witnesses non-termination of  $(\varphi, \vec{a})$ , then  $\hat{\eta}(\vec{c})$  witnesses non-termination of  $Tr_\eta(\varphi, \vec{a})$ . Here,  $\hat{\eta}: \mathcal{S}^d \rightarrow \mathcal{S}^d$  maps  $\vec{c}$  to  $\hat{\eta}(\vec{c}) = \tilde{c}(\eta(\vec{x})) = (\eta(\vec{x}))(\vec{c})$ .*

*Example 9 (Transforming Witnesses).* For the tuple  $\vec{c} = (5, 2)$  from [Ex. 4](#) and the automorphism  $\eta$  from [Ex. 3](#) with  $\eta(x_1) = x_2$  and  $\eta(x_2) = x_1 - x_2^2$ , we obtain

$$\hat{\eta}(\vec{c}) = (\eta(x_1), \eta(x_2))(\vec{c}) = (2, 5 - 2^2) = (2, 1).$$

As  $\vec{c} = (5, 2)$  witnesses non-termination of [Ex. 4](#),  $\hat{\eta}(\vec{c}) = (2, 1)$  witnesses non-termination of  $Tr_\eta(\varphi, \vec{a})$  due to [Lemma 8](#).

Finally, [Thm. 10](#) states that transforming a loop preserves (non-)termination.

**Theorem 10 (Tr Preserves Termination).** *If  $\eta \in \text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ , then  $(\varphi, \vec{a})$  terminates iff  $Tr_\eta(\varphi, \vec{a})$  terminates. Furthermore,  $\hat{\eta}$  is a bijection between the respective sets of witnesses for non-termination.*

Up to now, we only transformed a loop  $(\varphi, \vec{a})$  on  $\mathcal{S}^d$  using elements of  $\text{Aut}_{\mathcal{S}}(\mathcal{S}[\vec{x}])$ . However, we can also transform it into the loop  $Tr_\eta(\varphi, \vec{a})$  on  $\mathbb{R}_{\mathbb{A}}^d$  using an automorphism  $\eta \in \text{Aut}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$ . Nevertheless, our goal remains to prove termination on  $\mathcal{S}^d$  instead of  $\mathbb{R}_{\mathbb{A}}^d$ , which is not equivalent in general. Thus,

in Sect. 5 we will show how to analyze termination of loops on certain subsets  $F$  of  $\mathbb{R}_{\mathbb{A}}^d$ . This allows us to analyze termination of  $(\varphi, \vec{a})$  on  $\mathcal{S}^d$  by checking termination of  $Tr_{\eta}(\varphi, \vec{a})$  on the subset  $\widehat{\eta}(\mathcal{S}^d) \subseteq \mathbb{R}_{\mathbb{A}}^d$  instead.

By our definition of loops over a ring  $\mathcal{S}$ , we have  $\vec{a}(\vec{c}) \in \mathcal{S}^d$  for all  $\vec{c} \in \mathcal{S}^d$ , i.e.,  $\mathcal{S}^d$  is  $\vec{a}$ -invariant. This property is preserved by our transformation.

**Definition 11 ( $\vec{a}$ -Invariance).** *Let  $(\varphi, \vec{a})$  be a loop on  $\mathcal{S}^d$  and let  $F \subseteq \mathcal{S}^d$ . We call  $F$   $\vec{a}$ -invariant or update-invariant if for all  $\vec{c} \in F$  we have  $\vec{a}(\vec{c}) \in F$ .*

**Lemma 12.** *Let  $(\varphi, \vec{a})$  be a loop on  $\mathcal{S}^d$ , let  $F \subseteq \mathcal{S}^d$  be  $\vec{a}$ -invariant, and let  $\eta \in \text{Aut}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$ . Furthermore, let  $Tr_{\eta}(\varphi, \vec{a}) = (\varphi', \vec{a}')$ . Then  $\widehat{\eta}(F)$  is  $\vec{a}'$ -invariant.*

Recall that our goal is to reduce termination to a  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -formula. Clearly, termination on  $F$  cannot be encoded with such a formula if  $F$  cannot be defined via  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ . Thus, we require that  $F$  is  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable.

**Definition 13 ( $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -Definability).** *A set  $F \subseteq \mathbb{R}_{\mathbb{A}}^d$  is  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable if there is a  $\psi \in \text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  with free variables  $\vec{x}$  such that for all  $\vec{c} \in \mathbb{R}_{\mathbb{A}}^d$*

$$\vec{c} \in F \quad \text{iff} \quad \psi(\vec{c}) \text{ is valid.}$$

An example for a  $\text{Th}_{\exists}(\mathbb{Z}, \mathbb{R}_{\mathbb{A}})$ -definable set is  $\{(a, 0, a) \mid a \in \mathbb{Z}\}$ , which is characterized by the formula  $\exists a \in \mathbb{Z}. x_1 = a \wedge x_2 = 0 \wedge x_3 = a$ .

To analyze termination of  $(\varphi, \vec{a})$  on  $\mathcal{S}^d$ , we can analyze termination of  $Tr_{\eta}(\varphi, \vec{a})$  on  $\widehat{\eta}(\mathcal{S}^d) \subseteq \mathbb{R}_{\mathbb{A}}^d$  instead. The reason is that  $\vec{c} \in \mathcal{S}^d$  witnesses non-termination of  $(\varphi, \vec{a})$  iff  $\widehat{\eta}(\vec{c})$  witnesses non-termination of  $Tr_{\eta}(\varphi, \vec{a})$  due to Thm. 10, i.e.,  $\mathcal{S}^d$  contains a witness for non-termination of  $(\varphi, \vec{a})$  iff  $\widehat{\eta}(\mathcal{S}^d)$  contains a witness for non-termination of  $Tr_{\eta}(\varphi, \vec{a})$ . While  $\mathcal{S}^d$  is clearly  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable, the following lemma shows that  $\widehat{\eta}(\mathcal{S}^d)$  is  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable, too. More precisely,  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definability is preserved by polynomial endomorphisms.

**Lemma 14.** *Let  $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}_{\mathbb{A}}$  and let  $\eta \in \text{End}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$ . If  $F \subseteq \mathbb{R}_{\mathbb{A}}^d$  is  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable then so is  $\widehat{\eta}(F)$ .*

*Example 15.* The set  $\mathbb{Z}^2$  is  $\text{Th}_{\exists}(\mathbb{Z}, \mathbb{R}_{\mathbb{A}})$ -definable, as we have  $(x_1, x_2) \in \mathbb{Z}^2$  iff

$$\exists a, b \in \mathbb{Z}. x_1 = a \wedge x_2 = b.$$

Let  $\eta \in \text{End}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$  with  $\eta(x_1) = \frac{1}{2} \cdot x_1^2 + x_2^2$  and  $\eta(x_2) = x_2^2$ . Then  $\widehat{\eta}(\mathbb{Z}^2)$  is also  $\text{Th}_{\exists}(\mathbb{Z}, \mathbb{R}_{\mathbb{A}})$ -definable, because for  $x_1, x_2 \in \mathbb{R}_{\mathbb{A}}$ , we have  $(x_1, x_2) \in \widehat{\eta}(\mathbb{Z}^2)$  iff

$$\exists y_1, y_2 \in \mathbb{R}_{\mathbb{A}}, a, b \in \mathbb{Z}. y_1 = a \wedge y_2 = b \wedge x_1 = \frac{1}{2} \cdot y_1^2 + y_2^2 \wedge x_2 = y_2^2.$$

The following theorem shows that instead of regarding *solvable loops* [48], w.l.o.g. we can restrict ourselves to *twn-loops*. The reason is that every solvable loop with real eigenvalues can be transformed into a *twn-loop* by a *linear* automorphism  $\eta$ , i.e., the degree  $\text{deg}(\eta)$  of  $\eta$  is 1, where  $\text{deg}(\eta) = \max_{1 \leq i \leq d} \text{deg}(\eta(x_i))$ .

**Theorem 16.** *Let  $(\varphi, \vec{a})$  be a solvable loop with real eigenvalues. Then one can compute a linear automorphism  $\eta \in \text{Aut}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$  such that  $Tr_{\eta}(\varphi, \vec{a})$  is *twn*.*

We recapitulate our most important results on  $Tr$  in the following corollary.

**Corollary 17 (Properties of  $Tr$ ).** *Let  $(\varphi, \vec{a})$  be a loop,  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$ ,  $Tr_\eta(\varphi, \vec{a}) = (\varphi', \vec{a}')$ , and  $F \subseteq \mathcal{S}^d$  be  $\vec{a}$ -invariant and  $\text{Th}_\exists(\mathcal{S}, \mathbb{R}_\mathbb{A})$ -definable. Then*

1.  $\widehat{\eta}(F) \subseteq \mathbb{R}_\mathbb{A}^d$  is  $\vec{a}'$ -invariant and  $\text{Th}_\exists(\mathcal{S}, \mathbb{R}_\mathbb{A})$ -definable,
2.  $(\varphi, \vec{a})$  terminates on  $F$  iff  $(\varphi', \vec{a}')$  terminates on  $\widehat{\eta}(F)$ , and
3.  $\vec{c} \in F$  witnesses non-termination of  $(\varphi, \vec{a})$  iff  $\widehat{\eta}(\vec{c}) \in \widehat{\eta}(F)$  witnesses non-termination of  $(\varphi', \vec{a}')$ .

### 3.2 Finding Automorphisms to Transform Loops into *twn*-Form

The goal of the transformation from Sect. 3.1 is to transform  $(\varphi, \vec{a})$  into *twn*-form, such that termination of the resulting loop  $Tr_\eta(\varphi, \vec{a})$  can be analyzed by the technique which will be presented in Sect. 4 and 5. Hence, the remaining challenge is to find a suitable automorphism  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  such that  $Tr_\eta(\varphi, \vec{a})$  is *twn*. In this section, we will present two techniques to check the existence of such automorphisms *constructively*, i.e., these techniques can also be used to compute such automorphisms.

Note that the search for suitable automorphisms is closely related to the question if a polynomial endomorphism can be conjugated into a “de Jonquières”-automorphism, a difficult question from algebraic geometry [14]. So future advances in this field may help to improve the results of the current section.

The first technique (Thm. 20) reduces the search for a suitable automorphism of bounded degree to  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$ . It is known that for any automorphism the degree of its inverse has an upper bound in terms of the length  $d$  of  $\vec{x}$ , see [14, Cor. 2.3.4].

**Theorem 18.** *Let  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$ . Then we have  $\deg(\eta^{-1}) \leq (\deg(\eta))^{d-1}$ .*

By Thm. 18, checking if an endomorphism is indeed an automorphism can be reduced to  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$ . To do so, one encodes the existence of suitable coefficients of the polynomials  $\eta^{-1}(x_1), \dots, \eta^{-1}(x_d)$ , which all have at most degree  $(\deg(\eta))^{d-1}$ .

**Lemma 19.** *Let  $\eta \in \text{End}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$ . Then the question whether  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  holds is reducible to  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$ .*

Based on Lemma 19, we now present our first technique to find an automorphism  $\eta$  that transforms a loop into *twn*-form.

**Theorem 20 (Tr with Automorphisms of Bounded Degree).** *For any  $\delta \geq 0$ , the question whether there exists an  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  with  $\deg(\eta) \leq \delta$  such that  $Tr_\eta(\varphi, \vec{a})$  is *twn* is reducible to  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$ .*

So if the degree of  $\eta$  is bounded a priori, then it is decidable whether there exists an  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  such that  $Tr_\eta(\varphi, \vec{a})$  is *twn*, since  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$  is decidable.

We call a loop *twn-transformable* if there is an  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  such that  $Tr_\eta(\varphi, \vec{a})$  is *twn*. By Thm. 20, *twn*-transformability is *semi-decidable*, since one can increment  $\delta$  until a suitable automorphism is found. So in other words, any loop which is transformable to a *twn*-loop can be transformed via Thm. 20.

We call our transformation *Tr complete* for a class of loops if *every* loop from this class is *twn*-transformable. For such classes of loops, a suitable automorphism is *computable* by Thm. 20. Together with Thm. 16, we get the following corollary.



**Corollary 21.** *Tr is complete for solvable loops with real eigenvalues.*

Note that for solvable loops  $(\varphi, \vec{a})$ , instead of computing  $\eta$  using [Thm. 20](#), the proof of [Thm. 16](#) yields a more efficient way to compute a linear automorphism  $\eta$  such that  $Tr_\eta(\varphi, \vec{a})$  is *tw*. To this end, one computes the Jordan normal form of each  $A_i$  (see [Def. 2](#)), which is possible in polynomial time (see e.g., [[19, 46](#)]).

Our second technique to find suitable automorphisms for our transformation is restricted to *linear* automorphisms. In this case, it is decidable whether a loop can be transformed into a *tw*-loop  $(\varphi', \vec{a}')$  where the monomial for  $x_i$  has the coefficient 1 in each  $a'_i$ . The decision procedure checks whether a certain Jacobian matrix is *strongly nilpotent*, i.e., it is not based on a reduction to  $\text{Th}_\exists(\mathbb{R}_\mathbb{A})$ .

**Definition 22 (Strong Nilpotence).** *Let  $J \in (\mathbb{R}_\mathbb{A}[\vec{x}])^{d \times d}$  be a matrix of polynomials. For all  $1 \leq i \leq d$ , let  $\vec{y}^{(i)}$  be a vector of fresh variables.  $J$  is strongly nilpotent if  $\prod_{i=1}^d J[\vec{x}/\vec{y}^{(i)}] = 0^{d \times d}$ , where  $0^{d \times d} \in (\mathbb{R}_\mathbb{A}[\vec{x}])^{d \times d}$  is the zero matrix.*

Our second technique is formulated in the following theorem which follows from an existing result in linear algebra [[13, Thm. 1.6.](#)].

**Theorem 23 (Tr with Linear Automorphisms [[13, Thm. 1.6.](#)]).** *Let  $(\varphi, \vec{a})$  be a loop. The Jacobian matrix  $(\frac{\partial(a_i - x_i)}{\partial x_j})_{1 \leq i, j \leq d} \in (\mathbb{R}_\mathbb{A}[\vec{x}])^{d \times d}$  is strongly nilpotent iff there exists a linear automorphism  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  with*

$$Tr_\eta(\varphi, \vec{a}) = (\varphi', (x_1 + p_1, \dots, x_d + p_d)) \quad (2)$$

and  $p_i \in \mathbb{R}_\mathbb{A}[x_{i+1}, \dots, x_d]$ . Thus,  $Tr_\eta(\varphi, \vec{a})$  is *tw*.

As strong nilpotence of the Jacobian matrix is clearly decidable, [Thm. 23](#) gives rise to a decision procedure for the existence of a linear automorphism that transforms  $(\varphi, \vec{a})$  to the form [\(2\)](#).

*Example 24 (Finding Automorphisms).* The following loop on  $\mathcal{S}^3$  shows how our results enlarge the class of loops where termination is reducible to  $\text{Th}_\exists(\mathcal{S}, \mathbb{R}_\mathbb{A})$ .

$$\text{while } 4 \cdot x_2^2 + x_1 + x_2 + x_3 > 0 \text{ do } (x_1, x_2, x_3) \leftarrow (a_1, a_2, a_3) \quad (3)$$

$$\text{with } a_1 = x_1 + 8 \cdot x_1 \cdot x_2^2 + 16 \cdot x_2^3 + 16 \cdot x_2^2 \cdot x_3$$

$$a_2 = x_2 - x_1^2 - 4 \cdot x_1 \cdot x_2 - 4 \cdot x_1 \cdot x_3 - 4 \cdot x_2^2 - 8 \cdot x_2 \cdot x_3 - 4 \cdot x_3^2$$

$$a_3 = x_3 - 4 \cdot x_1 \cdot x_2^2 - 8 \cdot x_2^3 - 8 \cdot x_2^2 \cdot x_3 + x_1^2 + 4 \cdot x_1 \cdot x_2 +$$

$$4 \cdot x_1 \cdot x_3 + 4 \cdot x_2^2 + 8 \cdot x_2 \cdot x_3 + 4 \cdot x_3^2$$

It is clearly *not* in *tw*-form. To transform it, we use [Thm. 23](#). The Jacobian matrix  $J$  of  $(a_1 - x_1, a_2 - x_2, a_3 - x_3)$  is:

$$\begin{bmatrix} 8 \cdot x_2^2 & 16 \cdot x_1 \cdot x_2 + 48 \cdot x_2^2 + 32 \cdot x_2 \cdot x_3 & 16 \cdot x_2^2 \\ -2 \cdot x_1 - 4 \cdot x_2 - 4 \cdot x_3 & -4 \cdot x_1 - 8 \cdot x_2 - 8 \cdot x_3 & -4 \cdot x_1 - 8 \cdot x_2 - 8 \cdot x_3 \\ -4 \cdot x_2^2 + 2 \cdot x_1 + 4 \cdot x_2 + 4 \cdot x_3 & -8 \cdot x_1 \cdot x_2 - 24 \cdot x_2^2 - 16 \cdot x_2 \cdot x_3 + 4 \cdot x_1 + 8 \cdot x_2 + 8 \cdot x_3 & -8 \cdot x_2^2 + 4 \cdot x_1 + 8 \cdot x_2 + 8 \cdot x_3 \end{bmatrix}$$

One easily checks that  $J$  is strongly nilpotent. Thus, by [Thm. 23](#) the loop can be transformed into *tw*-form by a linear automorphism. Indeed, consider the linear automorphism  $\eta \in \text{Aut}_{\mathbb{R}_\mathbb{A}}(\mathbb{R}_\mathbb{A}[\vec{x}])$  induced by the matrix  $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ , i.e.,

$$x_1 \mapsto x_1 + x_2 + x_3, \quad x_2 \mapsto 2 \cdot x_2, \quad x_3 \mapsto x_1 + 2 \cdot x_2 + 2 \cdot x_3$$

with its inverse  $\eta^{-1}$

$$x_1 \mapsto 2 \cdot x_1 - x_3, \quad x_2 \mapsto \frac{1}{2} \cdot x_2, \quad x_3 \mapsto -x_1 - \frac{1}{2} \cdot x_2 + x_3.$$

If we transform our loop with  $\eta$ , we obtain the following *tw*n-loop:

$$\mathbf{while} \ x_1 + x_2^2 > 0 \ \mathbf{do} \ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 + x_2^2 \cdot x_3 \\ x_2 - 2 \cdot x_3^2 \\ x_3 \end{bmatrix} \quad (4)$$

If  $\mathcal{S} = \mathbb{R}_{\mathbb{A}}$ , then (4) terminates on  $\mathbb{R}_{\mathbb{A}}^3$  iff (3) terminates on  $\mathbb{R}_{\mathbb{A}}^3$  by [Thm. 10](#). Now assume  $\mathcal{S} = \mathbb{Z}$ , i.e., we are interested in termination of (3) on  $\mathbb{Z}^3$  instead of  $\mathbb{R}_{\mathbb{A}}^3$ . Note that  $\hat{\eta}$  maps  $\mathbb{Z}^3$  to the set of all  $\mathbb{Z}$ -linear combinations of columns of  $M$ , i.e.,

$$\hat{\eta}(\mathbb{Z}^3) = \{a \cdot (1, 0, 1) + b \cdot (1, 2, 2) + c \cdot (1, 0, 2) \mid a, b, c \in \mathbb{Z}\}.$$

By [Cor. 17](#), (4) terminates on  $\hat{\eta}(\mathbb{Z}^3)$  iff (3) terminates on  $\mathbb{Z}^3$ . Moreover,  $\hat{\eta}(\mathbb{Z}^3)$  is  $\text{Th}_{\exists}(\mathbb{Z}, \mathbb{R}_{\mathbb{A}})$ -definable: We have  $(x_1, x_2, x_3) \in \hat{\eta}(\mathbb{Z}^3)$  iff

$$\exists a, b, c \in \mathbb{Z}. \ x_1 = a \cdot 1 + b \cdot 1 + c \cdot 1 \wedge x_2 = b \cdot 2 \wedge x_3 = a \cdot 1 + b \cdot 2 + c \cdot 2.$$

In the following sections, we will see how to analyze termination of loops like (4) on sets that can be characterized by such formulas.

To summarize, if a loop is *tw*n-transformable, then we can *always* find a suitable automorphism via [Thm. 20](#). So whenever [Thm. 23](#) is applicable, a suitable linear automorphism can also be obtained by using [Thm. 20](#) for some fixed degree  $\delta \geq 1$ . Hence, our first technique from [Thm. 20](#) subsumes our second one from [Thm. 23](#). However, while [Thm. 20](#) is *always* applicable, [Thm. 23](#) is *easier* to apply. The reason is that for [Thm. 20](#) one has to check validity of a possibly *non-linear* formula over the reals, where the degree of the occurring polynomials depends on  $\delta$  and the update  $\vec{a}$  of the loop. So even when searching for a linear automorphism, one may obtain a non-linear formula if the loop is non-linear. In contrast, [Thm. 23](#) only requires linear algebra. So it is preferable to first check whether the loop can be transformed into a *tw*n-loop  $(\varphi', (x_1 + p_1, \dots, x_d + p_d))$  with  $x_i \notin \mathcal{V}(p_i)$  via a linear automorphism. This check is *decidable* due to [Thm. 23](#).

Note that the proof of [Thm. 20](#) and the proof of [[13](#), [Thm. 1.6.](#)] which implies [Thm. 23](#) are constructive. Thus, we can not only check the existence of a suitable automorphism, but we can also compute it whenever its existence can be proven.

## 4 Computing Closed Forms

Now we show how to reduce the termination problem of a *tw*n-loop on a  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable set to validity of a formula from  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ . Our reduction exploits that for *tw*n-loops  $(\varphi, \vec{a})$ , there is a closed form for the  $n$ -fold application of  $\vec{a}$  which can be represented as a vector of *poly-exponential expressions*.

As in [[15](#)], we restrict ourselves to *tn*n-loops (instead of *tw*n-loops), because each *tw*n-loop can be transformed into a *tn*n-loop via *chaining*.

**Definition 25 (Chaining).** Chaining a loop  $(\varphi, \vec{a})$  yields  $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ .

Clearly,  $(\varphi, \vec{a})$  terminates iff  $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$  terminates. Moreover, if  $(\varphi, \vec{a})$  is a *twn*-loop then  $(\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$  is a *tnn*-loop, i.e., the coefficient of each  $x_i$  in  $a_i(\vec{a})$  is non-negative. Thus, analogous to [15], we obtain the following theorem.

**Theorem 26.** *Termination of twn-loops is reducible to termination of tnn-loops.*

It is well known that closed forms for *tnn*-loops are computable, see, e.g., [28]. The reason is that the bodies of *tnn*-loops correspond to a special case of *C-finite recurrences*, which are known to be solvable [26]. The resulting closed forms may contain polynomial arithmetic and exponentiation w.r.t.  $n$  (as, e.g.,  $x_1 \leftarrow 2 \cdot x_1$  has the closed form  $x_1 \cdot 2^n$ ) as well as certain piecewise defined functions. For example, the closed form of  $x_1 \leftarrow 1$  is  $x_1^{(n)} = x_1$  if  $n = 0$  and  $x_1^{(n)} = 1$ , otherwise.

We use *poly-exponential expressions* [15]<sup>8</sup> to represent closed forms where piecewise defined functions are simulated via *characteristic functions*. Given a formula  $\psi$  over  $n$ , its characteristic function  $\llbracket \psi \rrbracket : \mathbb{N} \rightarrow \{0, 1\}$  evaluates to 1 iff  $\psi$  is satisfied (i.e.,  $\llbracket \psi \rrbracket(c) = 1$  if  $\psi[n/c]$  holds and  $\llbracket \psi \rrbracket(c) = 0$ , otherwise). In this way, we avoid handling piecewise defined functions via disjunctions (as done in the closed form of [28]). Poly-exponential expressions are sums of arithmetic terms over the variables  $\vec{x}$  and the additional designated variable  $n$ , where it is always clear which addend determines the asymptotic growth of the whole expression when increasing  $n$ . This is crucial for our reducibility proof in Sect. 5. In the following, for any set  $X \subseteq \mathbb{R}$ , any  $k \in X$ , and  $\triangleright \in \{\geq, >\}$ , let  $X_{\triangleright k} = \{x \in X \mid x \triangleright k\}$ .

**Definition 27 (Poly-Exponential Expressions).** *Let  $\mathcal{C}$  be the set of all finite conjunctions over  $\{n = c, n \neq c \mid c \in \mathbb{N}\}$  where  $n$  is a designated variable. The set of all poly-exponential expressions with the variables  $\vec{x}$  is*

$$\mathbb{PE}[\vec{x}] = \left\{ \sum_{j=1}^{\ell} \llbracket \psi_j \rrbracket \cdot \alpha_j \cdot n^{a_j} \cdot b_j^n \mid \ell, a_j \in \mathbb{N}, \psi_j \in \mathcal{C}, \alpha_j \in \mathbb{R}_{\mathbb{A}}[\vec{x}], b_j \in (\mathbb{R}_{\mathbb{A}})_{>0} \right\}.$$

So an example for a poly-exponential expression is

$$\llbracket n \neq 0 \wedge n \neq 1 \rrbracket \cdot \left( \frac{1}{2} \cdot x_1^2 + \frac{3}{4} \cdot x_2 - 1 \right) \cdot n^3 \cdot 3^n + \llbracket n = 1 \rrbracket \cdot (x_1 - x_2).$$

Note that the restriction to triangular loops ensures that the closed form does not contain complex numbers. For example, for arbitrary matrices  $A \in \mathcal{S}^{d \times d}$ , the update  $\vec{x} \leftarrow A \cdot \vec{x}$  is known to admit a closed form as in Def. 27 with complex  $b_j$ 's, whereas real numbers suffice for triangular matrices. Moreover, non-negativity is required to ensure  $b_j > 0$  (e.g., the non-*tnn* loop  $x_1 \leftarrow -x_1$  has the closed form  $x_1 \cdot (-1)^n$ ). So together with triangularity, weak non-linearity ensures that for every *tnn*-loop, one can compute a closed form  $\vec{q} \in (\mathbb{PE}[\vec{x}])^d$  with  $\vec{q} = \vec{a}^n$ .

*Example 28 (Closed Forms).* Reconsider the loop (4) from Ex. 24. This loop is *tnn* as  $\succ_{(4)} = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$  is well founded. Moreover, every variable  $x_i$  occurs with a non-negative coefficient in its corresponding update  $a_i$ . A closed form for the update after  $n \in \mathbb{N}$  loop iterations is:

<sup>8</sup> Our definition of poly-exponential expressions slightly generalizes [15, Def. 9] (e.g., we allow polynomials over the variables  $\vec{x}$  instead of just linear combinations).

$$\vec{q} = \begin{bmatrix} \frac{4}{3} \cdot x_3^5 \cdot n^3 + (-2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3) \cdot n^2 + (x_2^2 \cdot x_3 + \frac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3) \cdot n + x_1 \\ -2 \cdot x_3^2 \cdot n + x_2 \\ x_3 \end{bmatrix}$$

## 5 Reducing Termination of *tnn*-Loops to $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$

It is known that the bodies of *tnn*-loops can be linearized [39], i.e., one can reduce termination of a *tnn*-loop  $(\varphi, \vec{a})$  to termination of a linear-update *tnn*-loop  $(\varphi', \vec{a}')$  where  $\varphi'$  may be *non*-linear. Moreover, [55] showed decidability of termination for certain classes of conjunctive linear-update loops over  $\mathbb{R}$ , which cover conjunctive linear-update *tnn*-loops. So, by combining the results of [39] and [55], one can conclude that termination for *conjunctive tnn*-loops over  $\mathbb{R}$  is decidable.

However, we will now present a reduction of termination of *tnn*-loops to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  which applies to *tnn*-loops over *any* ring  $\mathbb{Z} \leq \mathcal{S} \leq \mathbb{R}$  and can handle also *disjunctions* in the loop condition. Moreover, our reduction yields tight complexity results on termination of linear loops over  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}_{\mathbb{A}}$ , and  $\mathbb{R}$ , and on termination of linear-update loops over  $\mathbb{R}_{\mathbb{A}}$  and  $\mathbb{R}$  (Sect. 6).

The idea of our reduction is similar to [15]. However, in [15], we considered conjunctive linear loops over  $\mathbb{Z}$ . In contrast, we now analyze termination of  $(\varphi, \vec{a})$  on an  $\vec{a}$ -invariant  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable subset of  $\mathbb{R}_{\mathbb{A}}^d$  and allow arbitrary propositional formulas and *non*-linearity in the condition. So the correctness proofs differ substantially from [15]. For reasons of space, we only show the major steps of our reduction and refer to [16] for more details.

In the following, let  $(\varphi, \vec{a})$  be *tnn*, let  $F \subseteq \mathbb{R}_{\mathbb{A}}^d$  be  $\vec{a}$ -invariant and  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable by the formula  $\psi_F$ , and let  $\vec{q} \in (\mathbb{P}\mathbb{E}[\vec{x}])^d$  be the closed form of  $\vec{a}^n$ .

We now show how to encode termination of  $(\varphi, \vec{a})$  on  $F$  into a  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -formula. More precisely, we show that there is a function with the following specification that is computable in polynomial time:

$$\begin{aligned} \text{Input : } & (\varphi, \vec{a}), \vec{q}, \text{ and } \psi_F \text{ as above} \\ \text{Result : } & \text{ a closed formula } \chi \in \text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}}) \text{ such that} \\ & \chi \text{ is valid iff } (\varphi, \vec{a}) \text{ does not terminate on } F \end{aligned} \quad (5)$$

We use the concept of *eventual non-termination* [8, 53], where the loop condition may be violated finitely often, i.e.,  $\vec{c}$  witnesses eventual non-termination of  $(\varphi, \vec{a})$  if  $\vec{a}^{n_0}(\vec{c})$  witnesses non-termination for some  $n_0 \in \mathbb{N}$ . Clearly,  $(\varphi, \vec{a})$  is non-terminating iff it is eventually non-terminating [41]. The formula  $\chi$  in (5) will encode the existence of a witness for eventual non-termination.

By the definition of  $\vec{q}$ ,  $(\varphi, \vec{a})$  is eventually non-terminating on  $F$  iff

$$\exists \vec{x} \in F, n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. \varphi(\vec{q}). \quad (6)$$

*Example 29.* Continuing Ex. 24 and 28, (4) is eventually non-terminating on

$$F = \widehat{\eta}(\mathbb{Z}^3) = \{a \cdot (1, 0, 1) + b \cdot (1, 2, 2) + c \cdot (1, 0, 2) \mid a, b, c \in \mathbb{Z}\}$$

iff there is a corresponding witness  $\vec{c} = (x_1, x_2, x_3)$ , i.e., iff

$$\exists x_1, x_2, x_3 \in F, n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. p > 0, \quad \text{where} \quad (7)$$

$$p = \left(\frac{4}{3} \cdot x_3^5\right) \cdot n^3 + \left(-2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4\right) \cdot n^2 \quad + \\ \left(x_2^2 \cdot x_3 + \frac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2\right) \cdot n \quad + \left(x_1 + x_2\right).$$

Let  $\vec{q}_{norm}$  be like  $\vec{q}$ , but each factor  $\llbracket \psi \rrbracket$  is replaced by 0 if it contains an equation and by 1, otherwise. The reason is that for large enough  $n$ , equations in  $\psi$  become false and negated equations become true. Thus, (6) is equivalent to

$$\exists \vec{x} \in F, n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. \varphi(\vec{q}_{norm}). \quad (8)$$

In this way, we obtain *normalized* poly-exponential expressions.

**Definition 30 (Normalized PEs).** We call  $p \in \mathbb{PE}[\vec{x}]$  normalized if it is in

$$\text{NPE}[\vec{x}] = \left\{ \sum_{j=1}^{\ell} \alpha_j \cdot n^{a_j} \cdot b_j^n \mid \ell, a_j \in \mathbb{N}, \alpha_j \in \mathbb{R}_{\mathbb{A}}[\vec{x}], b_j \in (\mathbb{R}_{\mathbb{A}})_{>0} \right\}.$$

*W.l.o.g., we always assume  $(b_i, a_i) \neq (b_j, a_j)$  if  $i \neq j$ . We define  $\text{NPE} = \text{NPE}[\emptyset]$ .*

As  $\varphi$  is a propositional formula over  $\mathbb{R}_{\mathbb{A}}[\vec{x}]$ -inequations,  $\varphi(\vec{q}_{norm})$  is a propositional formula over  $\text{NPE}[\vec{x}]$ -inequations. By (8), we need to check if there is an  $\vec{x} \in F$  such that  $\varphi(\vec{q}_{norm})$  is valid for large enough  $n$ . To do so, we generalize [15, Lemma 24]. As usual,  $g : \mathbb{N} \rightarrow \mathbb{R}$  dominates  $f : \mathbb{N} \rightarrow \mathbb{R}$  asymptotically ( $f \in o(g)$ ) if for all  $m > 0$  there is an  $n_0 \in \mathbb{N}$  such that  $|f(n)| < m \cdot |g(n)|$  for all  $n \in \mathbb{N}_{>n_0}$ .

**Lemma 31.** *Let  $b_1, b_2 \in (\mathbb{R}_{\mathbb{A}})_{>0}$  and  $a_1, a_2 \in \mathbb{N}$ . If  $(b_2, a_2) >_{lex} (b_1, a_1)$ , then  $n^{a_1} \cdot b_1^n \in o(n^{a_2} \cdot b_2^n)$ , where  $(b_2, a_2) >_{lex} (b_1, a_1)$  iff  $b_2 > b_1$  or  $b_2 = b_1 \wedge a_2 > a_1$ .*

In the following, let  $p \geq 0$  or  $p > 0$  occur in  $\varphi(\vec{q}_{norm})$ . Then we can order the coefficients of  $p$  according to the asymptotic growth of their addends w.r.t.  $n$ .

**Definition 32 (Ordering Coefficients).** Marked coefficients are of the form  $\alpha^{(b,a)}$  where  $\alpha \in \mathbb{R}_{\mathbb{A}}[\vec{x}]$ ,  $b \in (\mathbb{R}_{\mathbb{A}})_{>0}$ , and  $a \in \mathbb{N}$ . We define  $\text{unmark}(\alpha^{(b,a)}) = \alpha$  and  $\alpha_2^{(b_2, a_2)} \succ_{coef} \alpha_1^{(b_1, a_1)}$  if  $(b_2, a_2) >_{lex} (b_1, a_1)$ . Let  $p = \sum_{j=1}^{\ell} \alpha_j \cdot n^{a_j} \cdot b_j^n \in \text{NPE}[\vec{x}]$ , where  $\alpha_j \neq 0$  for all  $1 \leq j \leq \ell$ . Then the marked coefficients of  $p$  are  $\text{coefs}(p) = \{0^{(1,0)}\}$  if  $\ell = 0$  and  $\text{coefs}(p) = \{\alpha_j^{(b_j, a_j)} \mid 0 \leq j \leq \ell\}$ , otherwise.

*Example 33.* Continuing Ex. 29,  $\text{coefs}(p)$  is  $\{\alpha_1^{(1,3)}, \alpha_2^{(1,2)}, \alpha_3^{(1,1)}, \alpha_4^{(1,0)}\}$  where:

$$\alpha_1 = \frac{4}{3} \cdot x_3^5 \quad \alpha_2 = -2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4 \\ \alpha_3 = x_2^2 \cdot x_3 + \frac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2 \quad \alpha_4 = x_2^2 + x_1$$

Note that  $p(\vec{c}) \in \text{NPE}$  for any  $\vec{c} \in \mathbb{R}_{\mathbb{A}}^d$ , i.e., the only variable in  $p(\vec{c})$  is  $n$ . Now the  $\succ_{coef}$ -maximal addend determines the asymptotic growth of  $p(\vec{c})$ :

$$o(p(\vec{c})) = o(k \cdot n^a \cdot b^n) \quad \text{where } k^{(b,a)} = \max_{\succ_{coef}} (\text{coefs}(p(\vec{c}))). \quad (9)$$

Note that (9) would be incorrect for the case  $k = 0$  if we replaced  $o(p(\vec{c})) = o(k \cdot n^a \cdot b^n)$  with  $o(p(\vec{c})) = o(n^a \cdot b^n)$  as  $o(0) = \emptyset \neq o(1)$ . Obviously, (9) implies

$$\exists n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. \text{sign}(p(\vec{c})) = \text{sign}(k) \quad (10)$$

where  $\text{sign}(0) = 0$ ,  $\text{sign}(k) = 1$  if  $k > 0$ , and  $\text{sign}(k) = -1$  if  $k < 0$ . This already allows us to reduce eventual non-termination to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  if  $\varphi$  is an atom.

**Lemma 34.** *Given  $p \in \text{NPE}[\vec{x}]$  and  $\triangleright \in \{\geq, >\}$ , one can reduce validity of*

$$\exists \vec{x} \in F, n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. p \triangleright 0 \quad (11)$$

*to validity of a closed formula from  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  in polynomial time.<sup>9</sup>*

More precisely, (11) can be reduced to a formula  $\exists \vec{x} \in \mathbb{R}_{\mathbb{A}}^d. \psi_F \wedge \text{red}(p \triangleright 0)$ , where  $\text{red}(p \triangleright 0)$  is constructed as follows. By (10), we have  $p(\vec{c}) > 0$  for large enough values of  $n$  iff the coefficient of the asymptotically fastest-growing addend  $\alpha(\vec{c}) \cdot n^a \cdot b^n$  of  $p$  that does not vanish (i.e., where  $\alpha(\vec{c}) \neq 0$ ) is *positive*. Similarly, we have  $p(\vec{c}) < 0$  for large enough  $n$  iff  $\alpha(\vec{c}) < 0$ . If *all* addends of  $p$  vanish when instantiating  $\vec{x}$  with  $\vec{c}$ , then  $p(\vec{c}) = 0$ . In other words, (11) holds iff there is a  $\vec{c} \in F$  such that  $\text{unmark}(\max_{\succ_{\text{coef}}}(\text{coefs}(p(\vec{c})))) \triangleright 0$ . To express this in  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ , let  $\alpha_1, \dots, \alpha_{\ell}$  be the coefficients of  $p$ , ordered according to the asymptotic growth of the respective addends where  $\alpha_1$  belongs to the fastest-growing addend. Then

$$\begin{aligned} \text{red}(p > 0) & \text{ is } \bigvee_{j=1}^{\ell} \left( \alpha_j > 0 \wedge \bigwedge_{i=1}^{j-1} \alpha_i = 0 \right) \\ \text{and } \text{red}(p \geq 0) & \text{ is } \text{red}(p > 0) \vee \bigwedge_{i=1}^{\ell} \alpha_i = 0. \end{aligned}$$

Hence, (11) is equivalent to  $\exists \vec{x} \in \mathbb{R}_{\mathbb{A}}^d. \psi_F \wedge \text{red}(p \triangleright 0)$ .

*Example 35 (Reducing Eventual Non-Termination to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ ).* We finish Ex. 33 resp. 24 for  $\mathcal{S} = \mathbb{Z}$ , where  $\text{unmark}(\max_{\succ_{\text{coef}}}(\text{coefs}(p))) = \frac{4}{3} \cdot x_3^5$  and  $\psi_F$  is

$$\exists a, b, c \in \mathbb{Z}. x_1 = a + b + c \wedge x_2 = b \cdot 2 \wedge x_3 = a + b \cdot 2 + c \cdot 2.$$

Thus, (7) is valid iff  $\exists x_1, x_2, x_3 \in \mathbb{R}_{\mathbb{A}}. \psi_F \wedge \text{red}(p > 0)$  is valid where

$$\begin{aligned} \text{red}(p > 0) & = \alpha_1 > 0 \quad \vee (\alpha_2 > 0 \wedge \alpha_1 = 0) \\ & \vee (\alpha_3 > 0 \wedge \alpha_1 = \alpha_2 = 0) \vee (\alpha_4 > 0 \wedge \alpha_1 = \alpha_2 = \alpha_3 = 0). \end{aligned}$$

Then  $[x_1/1, x_2/0, x_3/1]$  satisfies  $\psi_F \wedge \alpha_1 > 0$  as  $(1, 0, 1) \in F$  (see Ex. 29) and  $(\frac{4}{3} \cdot x_3^5)[x_1/1, x_2/0, x_3/1] > 0$ . Thus,  $(1, 0, 1)$  witnesses eventual non-termination of (4). So the original loop (3) is non-terminating on  $\mathbb{Z}^3$  by Cor. 17 resp. Thm. 10.

Now we lift our reduction to propositional formulas. Note that a version of the following Thm. 36 that only covers conjunctions is a direct corollary of Lemma 34. To handle disjunctions, the proof of Thm. 36 exploits the crucial additional insight that a *tnn*-loop  $(\varphi \vee \varphi', \vec{a})$  terminates iff  $(\varphi, \vec{a})$  and  $(\varphi', \vec{a})$  terminate, which is not true in general (as, e.g., witnessed by the loop  $(x > 0 \vee -x > 0, -x)$ ).

**Theorem 36.** *Given a propositional formula  $\xi$  over the atoms  $\{p \triangleright 0 \mid p \in \text{NPE}[\vec{x}], \triangleright \in \{\geq, >\}\}$ , one can reduce validity of*

$$\exists \vec{x} \in F, n_0 \in \mathbb{N}. \forall n \in \mathbb{N}_{>n_0}. \xi \quad (12)$$

*to validity of a formula  $\exists \vec{x} \in \mathbb{R}_{\mathbb{A}}^d. \psi_F \wedge \text{red}(\xi) \in \text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$  in polynomial time.*

Here,  $\text{red}(\xi)$  results from replacing each atom  $p \triangleright 0$  in  $\xi$  by  $\text{red}(p \triangleright 0)$ .

Thm. 36 shows that the function (5) is computable (in polynomial time). This leads to the main result of this section.

<sup>9</sup> More precisely, the reduction of Lemma 34 and of the following Thm. 36 takes polynomially many steps in the size of the input of the function in (5).

**Theorem 37 (Reducing Termination).** *Termination of  $tnn$ -loops (resp.  $tw$ -loops) on  $\vec{a}$ -invariant and  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable sets is reducible to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ .*

However, in general this reduction is not computable in polynomial time, not even in exponential time. The reason is that closed forms  $\vec{q}$  of  $\vec{a}^n$  cannot be computed in exponential time if the update  $\vec{a}$  contains *non-linear* terms. For example, consider the following  $tnn$ -loop:

$$\mathbf{while\ true\ do\ } \vec{x} \leftarrow (d \cdot x_1, x_1^d, \dots, x_{d-2}^d, x_{d-1}^d) \quad (13)$$

The closed form for  $x_i^{(n)}$  is  $q_i = d^{(d^{i-1} \cdot (n-i+1))} \cdot x_1^{(d^{i-1})}$  for all  $n \geq d$ . Note that  $\log d^{(d^{d-1})}$  grows faster in  $d$  than any expression of the form  $c^d$ , where  $c \in \mathbb{N}$ . Thus, the closed form  $q_d \in \mathbb{PE}[\vec{x}]$  for  $x_d^{(n)}$  contains constants whose logarithm grows faster than any expression  $c^d$ . Hence,  $q_d$  cannot be computed in exponential time. As mentioned at the beginning of this section, the bodies of  $tnn$ -loops could also be linearized [39]. However, since the linearization of (13) contains these constants as well, it cannot be computed in exponential time, either.

Note that our reduction also works if  $\mathcal{S} = \mathbb{R}$ , i.e., termination over  $\mathbb{R}$  is reducible to  $\text{Th}_{\exists}(\mathbb{R}, \mathbb{R}_{\mathbb{A}})$ . As  $\mathbb{R}$  and  $\mathbb{R}_{\mathbb{A}}$  are *elementary equivalent*, i.e., a first-order formula is valid over  $\mathbb{R}$  iff it is valid over  $\mathbb{R}_{\mathbb{A}}$ , we get the following corollary.

**Corollary 38 ((Semi-)Decidability of (Non-)Termination).** *Let  $(\varphi, \vec{a})$  be a  $tw$ -loop and let  $F \subseteq \mathbb{R}_{\mathbb{A}}^d$  be  $\vec{a}$ -invariant and  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable.*

- (a) *The loop  $(\varphi, \vec{a})$  terminates over  $\mathbb{R}_{\mathbb{A}}$  iff it terminates over  $\mathbb{R}$ .*
- (b) *Termination of  $(\varphi, \vec{a})$  on  $F$  is decidable if  $\mathcal{S} = \mathbb{R}_{\mathbb{A}}$  or  $\mathcal{S} = \mathbb{R}$ .*
- (c) *Non-termination of  $(\varphi, \vec{a})$  on  $F$  is semi-decidable if  $\mathcal{S} = \mathbb{Z}$  or  $\mathcal{S} = \mathbb{Q}$ .*

Moreover, by Thm. 20 it is semi-decidable if a loop is *tw*-transformable. For conjunctive  $tw$ -loops, Cor. 38 (b) also follows from combining [39] and [55].

Our technique does not yield witnesses for non-termination, but the formula constructed by Thm. 36 describes *all* witnesses for *eventual* non-termination. So, the set of witnesses of *eventual* non-termination is  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ -definable whereas in general, the set of witnesses of non-termination is not (see [12]).

**Lemma 39.** *Let  $\xi = \varphi(\vec{q}_{norm})$ . Then  $\vec{c} \in \mathbb{R}_{\mathbb{A}}^d$  witnesses eventual non-termination of  $(\varphi, \vec{a})$  on  $F$  iff  $\psi_F(\vec{c}) \wedge (\text{red}(\xi))(\vec{c})$ .*

However, in [23] we showed how to compute witnesses for non-termination from witnesses for eventual non-termination of  $tw$ -loops. Thus, Lemma 39 combined with our results from [23] yields a technique to enumerate all witnesses for non-termination.

If  $(\varphi, \vec{a})$  results from the original loop by first transforming it into  $tw$ -form (Sect. 3) and by subsequently chaining it in order to obtain a loop in  $tnn$ -form (Sect. 4), then our approach can also be used to obtain witnesses for eventual non-termination of the original loop. In other words, one can compute a witness for the original loop from the witness for the transformed loop as in Cor. 17, since chaining clearly preserves witnesses for eventual non-termination. Alg. 1 summarizes our technique to check termination of  $tw$ -transformable-loops.

**Algorithm 1: Checking Termination**

**Input:** a *tw*n-transformable-loop  $(\varphi, \vec{a})$  and  $\psi_F \in \text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$   
**Result:**  $\top$  resp.  $\perp$  if (non-)termination of  $(\varphi, \vec{a})$  on  $F$  is proven, ? otherwise  
 $(\varphi, \vec{a}) \leftarrow \text{Tr}_{\eta}(\varphi, \vec{a})$ ,  $\psi_F \leftarrow \psi_{\tilde{\eta}(F)}$ , such that  $(\varphi, \vec{a})$  becomes *tw*n  
 $(\varphi, \vec{a}) \leftarrow (\varphi \wedge \varphi(\vec{a}), \vec{a}(\vec{a}))$ , such that  $(\varphi, \vec{a})$  becomes *tn*n  
 $\vec{q} \leftarrow$  closed form of  $\vec{a}^n$   
**if** (un)satisfiability of  $\psi_F \wedge \text{red}(\varphi(\vec{q}_{norm}))$  cannot be proven **then return** ?  
**if**  $\psi_F \wedge \text{red}(\varphi(\vec{q}_{norm}))$  is satisfiable **then return**  $\perp$  **else return**  $\top$

## 6 Complexity Analysis

We now analyze the complexity of our technique. We first regard *linear-update* loops, i.e., where the update is of the form  $\vec{x} \leftarrow A \cdot \vec{x} + \vec{b}$  with  $A \in \mathcal{S}^{d \times d}$  and  $\vec{b} \in \mathcal{S}^d$ . More precisely, we show that termination of linear loops with real spectrum is Co-NP-complete if  $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$  and that termination of linear-update loops with real spectrum is  $\forall\mathbb{R}$ -complete if  $\mathcal{S} = \mathbb{R}_{\mathbb{A}}$ . Since our proof is based on a reduction to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_{\mathbb{A}})$ , and  $\mathbb{R}_{\mathbb{A}}$  and  $\mathbb{R}$  are elementary equivalent, our results also hold if the program variables range over  $\mathbb{R}$ .

For our complexity results, we assume the usual dense encoding of univariate polynomials, i.e., a polynomial of degree  $k$  is represented as a list of  $k + 1$  coefficients. As discussed in [47], many problems which are considered to be efficiently solvable become intractable if polynomials are encoded sparsely (e.g., as lists of monomials where each monomial is a pair of its non-zero coefficient and its degree). With densely encoded polynomials, all common representations of algebraic numbers can be converted into each other in polynomial time [3].

When analyzing linear-update loops, w.l.o.g. we can assume  $\vec{b} = \vec{0}$ , since

$$\text{while } \varphi \text{ do } \vec{x} \leftarrow A \cdot \vec{x} + \vec{b} \quad \text{terminates iff} \quad (14)$$

$$\text{while } \varphi \wedge x_{\vec{b}} = 1 \text{ do } \begin{bmatrix} \vec{x} \\ x_{\vec{b}} \end{bmatrix} \leftarrow \begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \vec{x} \\ x_{\vec{b}} \end{bmatrix} \quad (15)$$

terminates, where  $x_{\vec{b}}$  is a fresh variable (see [24, 41]). Moreover,  $\vec{c}$  witnesses (eventual) non-termination for (14) iff  $\begin{bmatrix} \vec{c} \\ 1 \end{bmatrix}$  witnesses (eventual) non-termination for (15). Note that the only eigenvalue of  $\begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix}$  whose multiplicity increases in comparison to  $A$  is 1. Thus, to decide termination of linear-update loops with real spectrum, it suffices to decide termination of loops of the form  $(\varphi, A \cdot \vec{x})$  where  $A$  has only real eigenvalues.

Such loops can *always* be transformed into *tw*n-form using our transformation  $\text{Tr}$  from Sect. 3. To compute the required automorphism  $\eta$ , we compute the Jordan normal form  $Q$  of  $A$  together with the corresponding transformation matrix  $T$ , i.e.,  $T$  is an invertible real matrix such that  $A = T^{-1} \cdot Q \cdot T$ . Then  $Q$  is a triangular real matrix whose diagonal consists of the eigenvalues  $\lambda \in \mathbb{R}_{\mathbb{A}}$  of  $A$ . Now we define  $\eta \in \text{End}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$  by  $\eta(\vec{x}) = T \cdot \vec{x}$ . Then  $\eta \in \text{Aut}_{\mathbb{R}_{\mathbb{A}}}(\mathbb{R}_{\mathbb{A}}[\vec{x}])$  has the inverse  $\eta^{-1}(\vec{x}) = T^{-1} \cdot \vec{x}$ . Thus,  $\text{Tr}_{\eta}(\varphi, A \cdot \vec{x})$  is a *tw*n-loop with the update

$$(\eta(\vec{x})) (A \cdot \vec{x}) (\eta^{-1}(\vec{x})) = T \cdot A \cdot T^{-1} \cdot \vec{x} = Q \cdot \vec{x}.$$



The Jordan normal form  $Q$  as well as the matrices  $T$  and  $T^{-1}$  can be computed in polynomial time [19, 46]. Hence, we can decide whether all eigenvalues are real numbers in polynomial time by checking the diagonal entries of  $Q$ . Thus, we obtain the following lemma.

**Lemma 40.** *Let  $(\varphi, A \cdot \vec{x})$  be a linear-update loop.*

- (a) *It is decidable in polynomial time whether  $A$  has only real eigenvalues.*
- (b) *If  $A$  has only real eigenvalues, then we can compute a linear  $\eta \in \text{Aut}_{\mathbb{R}_A}(\mathbb{R}_A[\vec{x}])$  such that  $\text{Tr}_\eta(\varphi, A \cdot \vec{x})$  is a linear-update twin-loop in polynomial time.*
- (c) *If  $(\varphi, A \cdot \vec{x})$  is a linear loop, then so is  $\text{Tr}_\eta(\varphi, A \cdot \vec{x})$ .*

Hence, the transformation from Sect. 3 is complete for linear(-update) loops with real spectrum, i.e., every such loop can be transformed into a linear(-update) twin-loop. Note that the first part of Lemma 40 yields an efficient check whether a given linear(-update) loop has real spectrum.

As chaining (Def. 25) can clearly be done in polynomial time, w.l.o.g. we may assume that  $\text{Tr}_\eta(\varphi, A \cdot \vec{x}) = (\varphi', Q \cdot \vec{x})$  is *tnn*. Next, to analyze termination of a loop, our technique of Sect. 4 computes a closed form for the  $n$ -fold application of the update. For *tnn*-loops of the form  $(\varphi', Q \cdot \vec{x})$  where  $Q$  is a triangular matrix with non-negative diagonal entries, a suitable (i.e., poly-exponential) closed form can be computed in polynomial time analogously to [28, Prop. 5.2]. This closed form is linear in  $\vec{x}$ .

According to our approach in Sect. 5, we now proceed as in Alg. 1 and compute  $\text{red}(\varphi(\vec{q}_{\text{norm}})) \in \text{Th}_\exists(\mathcal{S}, \mathbb{R}_A)$ . The construction of this formula can be done in polynomial time due to Thm. 36. Hence, we get the following lemma.

**Lemma 41.** *Let  $(\varphi, A \cdot \vec{x})$  be a linear-update loop with real spectrum. Then we can compute a formula  $\psi \in \text{Th}_\exists(\mathcal{S}, \mathbb{R}_A)$  in polynomial time, such that  $\psi$  is valid iff the loop is non-terminating. If  $\varphi$  is linear, then so is  $\psi$ .*

Note that  $\psi$  is existentially quantified. Hence, if  $\varphi$  (and thus, also  $\psi$ ) is linear and  $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_A, \mathbb{R}\}$ , then invalidity of  $\psi$  is in Co-NP as validity of such formulas is in NP [42]. Thus, we obtain the first main result of this section.

**Theorem 42 (Co-NP-Completeness).** *Termination of linear loops  $(\varphi, A \cdot \vec{x} + \vec{b})$  with real spectrum over  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}_A$ , and  $\mathbb{R}$  is Co-NP-complete.*

For Co-NP-hardness, let  $\xi$  be a propositional formula over the variables  $\vec{x}$ . Then  $(\xi[x_i/(x_i > 0) \mid 1 \leq i \leq d], \vec{x})$  terminates iff  $\xi$  is unsatisfiable. So Co-NP-hardness follows from Co-NP-hardness of unsatisfiability of propositional formulas.

We now consider linear-update loops with real spectrum (and possibly non-linear loop conditions) on  $\mathbb{R}_A^d$  and  $\mathbb{R}^d$ . Here, non-termination is  $\exists\mathbb{R}$ -complete.

**Definition 43** ( $\exists\mathbb{R}$  [50, 51]). *Let  $\text{Th}_\exists(\mathbb{R})_\top = \{\psi \in \text{Th}_\exists(\mathbb{R}) \mid \psi \text{ is satisfiable}\}$ . The complexity class  $\exists\mathbb{R}$  is the closure of  $\text{Th}_\exists(\mathbb{R})_\top$  under poly-time-reductions.*

We have  $\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$ . By Lemma 41, non-termination of linear-update loops with real spectrum is in  $\exists\mathbb{R}$ . It is also  $\exists\mathbb{R}$ -hard since  $(\varphi, \vec{x})$  is non-terminating iff  $\varphi$  is satisfiable. So non-termination is  $\exists\mathbb{R}$ -complete, i.e., termination is Co- $\exists\mathbb{R}$ -complete (where Co- $\exists\mathbb{R} = \forall\mathbb{R}$  [51]).

**Theorem 44 ( $\forall\mathbb{R}$ -Completeness).** *Termination of linear-update loops with real spectrum over  $\mathbb{R}_A$  and  $\mathbb{R}$  is  $\forall\mathbb{R}$ -complete.*

Recall that the bodies of *tnn*-loops can be linearized [39]. The loop (13) showed that in general, this linearization is not computable in exponential time. However, if the degree of the polynomials in the update is bounded by a constant  $M$ , or if the number of variables  $d$  is bounded by a constant  $D$ , then the linearization is in EXPTIME (see the proof of Thm. 45 in [16]). Combining this with Thm. 44 yields Thm. 45 (a). If the number of variables is bounded, then checking validity of an existential formula over the reals is in P (see [2]). So in this case, combining the fact that linearization is in EXPTIME with Lemma 41 yields Thm. 45 (b).

**Theorem 45.** *Let  $M, D \in \mathbb{N}$  be fixed. Termination of *tnn*-loops over  $\mathbb{R}_A$  and  $\mathbb{R}$*

- (a) *is in 2-EXPTIME if the maximal degree in the update is at most  $M$ .*
- (b) *is in EXPTIME if the number of variables is at most  $D$ .*

## 7 Related Work and Conclusion

We presented a reduction from termination of *tnn*-loops to  $\text{Th}_{\exists}(\mathcal{S}, \mathbb{R}_A)$ . This implies decidability of termination over  $\mathcal{S} = \mathbb{R}_A$  and  $\mathcal{S} = \mathbb{R}$  and semi-decidability of non-termination over  $\mathcal{S} = \mathbb{Z}$  and  $\mathcal{S} = \mathbb{Q}$ . Moreover, we showed how to transform certain non-*tnn*-loops into *tnn*-form, which generalizes our results to a wider class of loops, including *solvable loops with real eigenvalues*. We also showed that *tnn*-transformability is semi-decidable. Finally, we used our results to prove Co-NP-completeness (resp.  $\forall\mathbb{R}$ -completeness) of termination of linear (resp. linear-update) loops with real spectrum.

*Related Work:* In contrast to *automated* termination analysis (e.g., [1, 4, 5, 7, 9, 10, 17, 20, 21, 22, 30, 31, 32, 33, 43]), we investigate *decidability* of termination for certain classes of loops. Clearly, such decidability results can only be obtained for quite restricted classes of programs.

Nevertheless, many techniques used in automated tools for termination analysis (e.g., the application of ranking functions) focus on similar classes of loops, since such loops occur as sub-programs in (abstractions of) real programs. Tools based on these techniques have turned out to be very successful, also for larger classes of programs. Thus, these tools could benefit from integrating our (semi-)decision procedures and applying them instead of incomplete techniques for any sub-program that can be transformed into a *tnn*-loop.

Related work on decidability of termination also considers related (and often more restricted) classes of loops. For linear conjunctive loops (where the loop condition is a conjunction), termination over  $\mathbb{R}$  [34, 37, 53],  $\mathbb{Q}$  [8], and  $\mathbb{Z}$  [24] is decidable. Tiwari [53] uses the special case of our *tnn*-transformation from Sect. 6 where the loop and the automorphism are linear. In contrast to these techniques, our approach applies to *non-linear* loops with *arbitrary* loop conditions over *various rings*.

*Linearization* is an alternative attempt to handle non-linearity. While the *update* of solvable loops can be linearized [39, 49], the *guard* cannot. Otherwise, one could linearize any loop ( $p = 0, \vec{x}$ ), which terminates over  $\mathbb{Z}$  iff  $p$  has no integer root. With [24], this would imply decidability of Hilbert’s Tenth Problem.

Regarding complexity, [41] proves that termination of conjunctive linear loops over  $\mathbb{Z}$  with update  $\vec{x} \leftarrow A \cdot \vec{x} + \vec{b}$  is in EXPSPACE if  $A$  is diagonalizable resp. in PSPACE if  $|\vec{x}| \leq 4$ . Moreover, [41] states that the techniques from [8, 53] run in polynomial time. So termination of conjunctive linear loops over  $\mathbb{Q}$  and  $\mathbb{R}$  is in P.

Our Co-NP-completeness result is orthogonal to those results as we allow disjunctions in the loop condition. Moreover, Co-NP-completeness also holds for termination over  $\mathbb{Z}$ , whereas [8, 53] only consider termination over  $\mathbb{Q}$  resp.  $\mathbb{R}$ .

In the non-linear case, [35] proves decidability of termination for conjunctive loops on  $\mathbb{R}^d$  for the case that the loop condition defines a compact and connected subset of  $\mathbb{R}^d$ . In [55], decidability of termination of conjunctive linear-update loops on  $\mathbb{R}^d$  with the *non-zero minimum property* is shown, which covers conjunctive linear-update loops with real spectrum. In combination with [39], this yields a decision procedure for termination of conjunctive *turn*-loops over  $\mathbb{R}$ . For general conjunctive linear-update loops on  $\mathbb{R}^d$  undecidability is conjectured. Furthermore, [38] proves that termination of (not necessarily conjunctive) linear-update loops is decidable if the loop condition describes a compact set. Finally, [56] gives sufficient criteria for (non-)termination of solvable loops and [36] introduces sufficient conditions under which termination of non-deterministic non-linear loops on  $\mathbb{R}^d$  can be reduced to satisfiability of a semi-algebraic system.

For linear-update loops with real spectrum over  $\mathbb{R}$ , we prove  $\forall\mathbb{R}$ -completeness of termination, whereas [55] does not give tight complexity results. The approach from [56] is incomplete, whereas we present a complete reduction from termination to the respective first-order theory. The work in [36] is orthogonal to ours as it only applies to loops that satisfy certain non-trivial conditions. Moreover, we consider loops with arbitrary loop conditions over various rings, whereas [35, 36, 55] only consider conjunctive loops over  $\mathbb{R}$  and [38] only considers loops over  $\mathbb{R}$  where the loop condition defines a compact set.

Finally, several other works exploit the existence of closed forms for solvable (or similar classes of) loops to, e.g., analyze termination for *fixed* inputs or deduce invariants (e.g., [23, 25, 27, 28, 29, 39, 40, 48, 49]). While our approach covers solvable loops with real eigenvalues (by Cor. 21), it also applies to loops which are not solvable, see Ex. 24. Note that our transformation from Sect. 3 may also be of interest for other techniques for solvable or other sub-classes of polynomial loops, as it may be used to extend the applicability of such approaches.

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