# A Dependency Pair Framework for Relative Termination of Term Rewriting

**Jürgen Giesl**

joint work with **Jan-Christoph Kassing** and **Grigory Vartanyan**

LuFG Programming Languages & Verification, RWTH Aachen University, Germany

WST, September 2025

# Relative Termination of TRSs

$24/[4,3]$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

$$24/[4,3]$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

$$24/[4,3] \to_{\mathcal{B}_{\mathsf{com}}} 24/[3,4]$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

$$24/[4,3] \to_{\mathcal{B}_{\mathsf{com}}} 24/[3,4] \to^{+}_{\mathcal{R}_{\mathsf{divL}}} 8/[4]$$

# Relative Termination of TRSs

$$24/[4,3] \;=\; (24/4)/3 \;=\; 2 \;=\; (24/3)/4 \;=\; 24/[3,4]$$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

$$24/[4,3] \to_{\mathcal{B}_{\mathsf{com}}} 24/[3,4] \to^{+}_{\mathcal{R}_{\mathsf{divL}}} 8/[4]$$

**Relative Termination**

$\mathcal{R}/\mathcal{B}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} \circ \to^{*}_{\mathcal{B}} t_1 \to_{\mathcal{R}} \circ \to^{*}_{\mathcal{B}} \ldots$

# Dependency Pairs

$$
\begin{array}{rrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x, 0) & \to & x \\
& \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
& \mathsf{div}(0, \mathsf{s}(y)) & \to & 0 \\
& \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x, y), \mathsf{s}(y))) \\
\\
& \mathsf{divL}(x, \mathsf{nil}) & \to & x \\
& \mathsf{divL}(x, \mathsf{cons}(y, xs)) & \to & \mathsf{divL}(\mathsf{div}(x, y), xs)
\end{array}
$$

# Dependency Pairs

$$
\begin{array}{rrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x, 0) & \to & x \\
& \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
& \mathsf{div}(0, \mathsf{s}(y)) & \to & 0 \\
& \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x, y), \mathsf{s}(y))) \\
& & & \\
& \mathsf{divL}(x, \mathsf{nil}) & \to & x \\
& \mathsf{divL}(x, \mathsf{cons}(y, xs)) & \to & \mathsf{divL}(\mathsf{div}(x, y), xs)
\end{array}
$$

**Defined Symbols**: minus, div, divL

# Dependency Pairs

$$
\begin{array}{lrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x, 0) & \to & x \\
& \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
& \mathsf{div}(0, \mathsf{s}(y)) & \to & 0 \\
& \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x, y), \mathsf{s}(y))) \\
& & & \\
& \mathsf{divL}(x, \mathsf{nil}) & \to & x \\
& \mathsf{divL}(x, \mathsf{cons}(y, xs)) & \to & \mathsf{divL}(\mathsf{div}(x, y), xs)
\end{array}
$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

# Dependency Pairs

$$
\begin{array}{rrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x, 0) & \to & x \\
& \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\
& \mathsf{div}(0, \mathsf{s}(y)) & \to & 0 \\
& \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x, y), \mathsf{s}(y))) \\
& & & \\
& \mathsf{divL}(x, \mathsf{nil}) & \to & x \\
& \mathsf{divL}(x, \mathsf{cons}(y, xs)) & \to & \mathsf{divL}(\mathsf{div}(x, y), xs)
\end{array}
$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1, \ldots, \ell_n) \to \ldots g(r_1, \ldots, r_m) \ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1, \ldots, \ell_n) \to g^{\#}(r_1, \ldots, r_m)$ is a dependency pair.

# Dependency Pairs

$\mathcal{DP}(\mathcal{R}_{\text{divL}})$ : $\text{minus}^{\#}(\text{s}(x), \text{s}(y)) \rightarrow \text{minus}^{\#}(x, y)$

$$
\begin{array}{rrcl}
\mathcal{R}_{\text{divL}}: & \text{minus}(x, 0) & \rightarrow & x \\
& \text{minus}(\text{s}(x), \text{s}(y)) & \rightarrow & \text{minus}(x, y) \\
& \text{div}(0, \text{s}(y)) & \rightarrow & 0 \\
& \text{div}(\text{s}(x), \text{s}(y)) & \rightarrow & \text{s}(\text{div}(\text{minus}(x, y), \text{s}(y))) \\
& & & \\
& \text{divL}(x, \text{nil}) & \rightarrow & x \\
& \text{divL}(x, \text{cons}(y, xs)) & \rightarrow & \text{divL}(\text{div}(x, y), xs)
\end{array}
$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1, \ldots, \ell_n) \rightarrow \ldots g(r_1, \ldots, r_m) \ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1, \ldots, \ell_n) \rightarrow g^{\#}(r_1, \ldots, r_m)$ is a dependency pair.

# Dependency Pairs

$$\mathcal{DP}(\mathcal{R}_{\text{divL}}):\quad \begin{aligned} \text{minus}^{\#}(\text{s}(x),\text{s}(y)) &\to \text{minus}^{\#}(x,y)\\ \text{div}^{\#}(\text{s}(x),\text{s}(y)) &\to \text{minus}^{\#}(x,y)\end{aligned}$$

$$\mathcal{R}_{\text{divL}}:\quad \begin{aligned} \text{minus}(x,0) &\to x\\ \text{minus}(\text{s}(x),\text{s}(y)) &\to \text{minus}(x,y)\\ \text{div}(0,\text{s}(y)) &\to 0\\ \text{div}(\text{s}(x),\text{s}(y)) &\to \text{s}(\text{div}(\text{minus}(x,y),\text{s}(y)))\\ \\ \text{divL}(x,\text{nil}) &\to x\\ \text{divL}(x,\text{cons}(y,xs)) &\to \text{divL}(\text{div}(x,y),xs)\end{aligned}$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1,\ldots,\ell_n) \to \ldots g(r_1,\ldots,r_m)\ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1,\ldots,\ell_n) \to g^{\#}(r_1,\ldots,r_m)$ is a dependency pair.

# Dependency Pairs

$$\begin{aligned}\mathcal{DP}(\mathcal{R}_{\text{divL}}):\ \text{minus}^{\#}(\text{s}(x),\text{s}(y)) &\to \text{minus}^{\#}(x,y)\\ \text{div}^{\#}(\text{s}(x),\text{s}(y)) &\to \text{minus}^{\#}(x,y)\\ \text{div}^{\#}(\text{s}(x),\text{s}(y)) &\to \text{div}^{\#}(\text{minus}(x,y),\text{s}(y))\end{aligned}$$

$$\begin{aligned}\mathcal{R}_{\text{divL}}:\ \text{minus}(x,0) &\to x\\ \text{minus}(\text{s}(x),\text{s}(y)) &\to \text{minus}(x,y)\\ \text{div}(0,\text{s}(y)) &\to 0\\ \text{div}(\text{s}(x),\text{s}(y)) &\to \text{s}(\text{div}(\text{minus}(x,y),\text{s}(y)))\\ \\ \text{divL}(x,\text{nil}) &\to x\\ \text{divL}(x,\text{cons}(y,xs)) &\to \text{divL}(\text{div}(x,y),xs)\end{aligned}$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1,\ldots,\ell_n) \to \ldots g(r_1,\ldots,r_m)\ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1,\ldots,\ell_n) \to g^{\#}(r_1,\ldots,r_m)$ is a dependency pair.

# Dependency Pairs

$$\begin{array}{lrcl} \mathcal{DP}(\mathcal{R}_{\text{divL}}): & \text{minus}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{minus}^{\#}(x,y) \\ & \text{div}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{minus}^{\#}(x,y) \\ & \text{div}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{div}^{\#}(\text{minus}(x,y),\text{s}(y)) \\ & \text{divL}^{\#}(x,\text{cons}(y,xs)) & \to & \text{div}^{\#}(x,y) \end{array}$$

$$\begin{array}{lrcl} \mathcal{R}_{\text{divL}}: & \text{minus}(x,0) & \to & x \\ & \text{minus}(\text{s}(x),\text{s}(y)) & \to & \text{minus}(x,y) \\ & \text{div}(0,\text{s}(y)) & \to & 0 \\ & \text{div}(\text{s}(x),\text{s}(y)) & \to & \text{s}(\text{div}(\text{minus}(x,y),\text{s}(y))) \\ & \text{divL}(x,\text{nil}) & \to & x \\ & \text{divL}(x,\text{cons}(y,xs)) & \to & \text{divL}(\text{div}(x,y),xs) \end{array}$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1,\ldots,\ell_n) \to \ldots g(r_1,\ldots,r_m)\ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1,\ldots,\ell_n) \to g^{\#}(r_1,\ldots,r_m)$ is a dependency pair.

# Dependency Pairs

$$\begin{array}{lrcl}
\mathcal{DP}(\mathcal{R}_{\text{divL}}): & \text{minus}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{minus}^{\#}(x,y)\\
& \text{div}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{minus}^{\#}(x,y)\\
& \text{div}^{\#}(\text{s}(x),\text{s}(y)) & \to & \text{div}^{\#}(\text{minus}(x,y),\text{s}(y))\\
& & & \\
& \text{divL}^{\#}(x,\text{cons}(y,xs)) & \to & \text{div}^{\#}(x,y)\\
& \text{divL}^{\#}(x,\text{cons}(y,xs)) & \to & \text{divL}^{\#}(\text{div}(x,y),xs)
\end{array}$$

$$\begin{array}{lrcl}
\mathcal{R}_{\text{divL}}: & \text{minus}(x,0) & \to & x\\
& \text{minus}(\text{s}(x),\text{s}(y)) & \to & \text{minus}(x,y)\\
& \text{div}(0,\text{s}(y)) & \to & 0\\
& \text{div}(\text{s}(x),\text{s}(y)) & \to & \text{s}(\text{div}(\text{minus}(x,y),\text{s}(y)))\\
& & & \\
& \text{divL}(x,\text{nil}) & \to & x\\
& \text{divL}(x,\text{cons}(y,xs)) & \to & \text{divL}(\text{div}(x,y),xs)
\end{array}$$

**Defined Symbols**: minus, div, divL

**Constructor Symbols**: s, 0, cons, nil

**Dependency Pairs**

If $f(\ell_1,\ldots,\ell_n) \to \ldots g(r_1,\ldots,r_m)\ldots$ is a rule and $g$ is defined, then $f^{\#}(\ell_1,\ldots,\ell_n) \to g^{\#}(r_1,\ldots,r_m)$ is a dependency pair.

# Dependency Pairs

$$\begin{array}{lrcl}
\mathcal{DP}(\mathcal{R}_{\mathsf{divL}}): & \mathsf{minus}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))\\
\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{div}^{\#}(x,y)\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs)
\end{array}$$

$$\begin{array}{lrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x,0) & \to & x\\
& \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}(x,y)\\
& \mathsf{div}(0,\mathsf{s}(y)) & \to & 0\\
& \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
& \mathsf{divL}(x,\mathsf{nil}) & \to & x\\
& \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{array}$$

**Termination of $(\mathcal{D},\mathcal{R})$**

$(\mathcal{D},\mathcal{R})$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} t_1 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} \ldots$

# Dependency Pairs

$$\begin{array}{lrcl}
\mathcal{DP}(\mathcal{R}_{\mathsf{divL}}): & \mathsf{minus}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))\\
\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{div}^{\#}(x,y)\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs)
\end{array}$$

$$\begin{array}{lrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x,0) & \to & x\\
& \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}(x,y)\\
& \mathsf{div}(0,\mathsf{s}(y)) & \to & 0\\
& \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
& \mathsf{divL}(x,\mathsf{nil}) & \to & x\\
& \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{array}$$

## Termination of $(\mathcal{D},\mathcal{R})$

$(\mathcal{D},\mathcal{R})$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} t_1 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} \ldots$

## Reminder: Relative Termination of $\mathcal{R}/\mathcal{B}$

$\mathcal{R}/\mathcal{B}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} t_1 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} \ldots$

# Dependency Pairs

$$\begin{array}{rrcl}
\mathcal{DP}(\mathcal{R}_{\mathsf{divL}}): & \mathsf{minus}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))\\
& & & \\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{div}^{\#}(x,y)\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs)
\end{array}$$

$$\begin{array}{rrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x,0) & \to & x\\
& \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}(x,y)\\
& \mathsf{div}(0,\mathsf{s}(y)) & \to & 0\\
& \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
& & & \\
& \mathsf{divL}(x,\mathsf{nil}) & \to & x\\
& \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{array}$$

## Termination of $(\mathcal{D},\mathcal{R})$

$(\mathcal{D},\mathcal{R})$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} t_1 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} \ldots$

## Reminder: Relative Termination of $\mathcal{R}/\mathcal{B}$

$\mathcal{R}/\mathcal{B}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} t_1 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} \ldots$

## **Theorem**: Chain Criterion [Arts & Giesl 2000]

$\mathcal{R}$ is terminating iff $(\mathcal{DP}(\mathcal{R}),\mathcal{R})$ is terminating

# Dependency Pairs

$$\begin{array}{lrcl}
\mathcal{DP}(\mathcal{R}_{\mathsf{divL}}): & \mathsf{minus}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}^{\#}(x,y)\\
& \mathsf{div}^{\#}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))\\
& & & \\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{div}^{\#}(x,y)\\
& \mathsf{divL}^{\#}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs)
\end{array}$$

$$\begin{array}{lrcl}
\mathcal{R}_{\mathsf{divL}}: & \mathsf{minus}(x,0) & \to & x\\
& \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}(x,y)\\
& \mathsf{div}(0,\mathsf{s}(y)) & \to & 0\\
& \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
& & & \\
& \mathsf{divL}(x,\mathsf{nil}) & \to & x\\
& \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to & \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{array}$$

## Termination of $(\mathcal{D},\mathcal{R})$

$(\mathcal{D},\mathcal{R})$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} t_1 \to_{\mathcal{D}} \circ \to_{\mathcal{R}}^{*} \ldots$

## Reminder: Relative Termination of $\mathcal{R}/\mathcal{B}$

$\mathcal{R}/\mathcal{B}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} t_1 \to_{\mathcal{R}} \circ \to_{\mathcal{B}}^{*} \ldots$

## **Theorem**: Chain Criterion [Arts & Giesl 2000]

$\mathcal{R}$ is terminating iff $\mathcal{DP}(\mathcal{R})/\mathcal{R}$ is terminating

# Dependency Pair Framework

- Key idea:
  - Transform a “large” problem into simpler sub-problems (divide and conquer)

# Dependency Pair Framework

- Key idea:
  - Transform a "large" problem into simpler sub-problems (divide and conquer)
- Objects that we work with:
  - DP problems $(\mathcal{D}, \mathcal{R})$ with a set of DPs $\mathcal{D}$ and a TRS $\mathcal{R}$

# Dependency Pair Framework

- Key idea:
  - Transform a "large" problem into simpler sub-problems (divide and conquer)
- Objects that we work with:
  - DP problems $(\mathcal{D}, \mathcal{R})$ with a set of DPs $\mathcal{D}$ and a TRS $\mathcal{R}$
- Initial DP problem for TRS $\mathcal{R}$:
  - (Chain Criterion) Use all dependency pairs and all rules: $(\mathcal{DP}(\mathcal{R}), \mathcal{R})$

# Dependency Pair Framework

- Key idea:
  - Transform a "large" problem into simpler sub-problems (divide and conquer)
- Objects that we work with:
  - DP problems $(\mathcal{D}, \mathcal{R})$ with a set of DPs $\mathcal{D}$ and a TRS $\mathcal{R}$
- Initial DP problem for TRS $\mathcal{R}$:
  - (Chain Criterion) Use all dependency pairs and all rules: $(\mathcal{DP}(\mathcal{R}), \mathcal{R})$
- Creating smaller problems:
  - DP processors: $\mathit{Proc}(\mathcal{D}, \mathcal{R}) = \{(\mathcal{D}_1, \mathcal{R}_1), \ldots, (\mathcal{D}_k, \mathcal{R}_k)\}$

# Dependency Pair Framework

- Key idea:
  - Transform a "large" problem into simpler sub-problems (divide and conquer)
- Objects that we work with:
  - DP problems $(\mathcal{D}, \mathcal{R})$ with a set of DPs $\mathcal{D}$ and a TRS $\mathcal{R}$
- Initial DP problem for TRS $\mathcal{R}$:
  - (Chain Criterion) Use all dependency pairs and all rules: $(\mathcal{DP}(\mathcal{R}), \mathcal{R})$
- Creating smaller problems:
  - DP processors: $\mathit{Proc}(\mathcal{D}, \mathcal{R}) = \{(\mathcal{D}_1, \mathcal{R}_1), \ldots, (\mathcal{D}_k, \mathcal{R}_k)\}$
  - $\mathit{Proc}$ is sound: if all $(\mathcal{D}_i, \mathcal{R}_i)$ are terminating, then $(\mathcal{D}, \mathcal{R})$ is terminating

# Dependency Pair Framework

- Key idea:
  - Transform a "large" problem into simpler sub-problems (divide and conquer)
- Objects that we work with:
  - DP problems $(\mathcal{D}, \mathcal{R})$ with a set of DPs $\mathcal{D}$ and a TRS $\mathcal{R}$
- Initial DP problem for TRS $\mathcal{R}$:
  - (Chain Criterion) Use all dependency pairs and all rules: $(\mathcal{DP}(\mathcal{R}), \mathcal{R})$
- Creating smaller problems:
  - DP processors: $\mathit{Proc}(\mathcal{D}, \mathcal{R}) = \{(\mathcal{D}_1, \mathcal{R}_1), \ldots, (\mathcal{D}_k, \mathcal{R}_k)\}$
  - *Proc* is sound: if all $(\mathcal{D}_i, \mathcal{R}_i)$ are terminating, then $(\mathcal{D}, \mathcal{R})$ is terminating
  - *Proc* is complete: if $(\mathcal{D}, \mathcal{R})$ is terminating, then all $(\mathcal{D}_i, \mathcal{R}_i)$ are terminating

# Timeline


- 2000: DPs for termination [Arts & Giesl 2000, . . . ]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"
- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $(\mathcal{DP}(\mathcal{R}), \mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Timeline


- 2000: DPs for termination [Arts & Giesl 2000, . . . ]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"

- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $(\mathcal{DP}(\mathcal{R}), \mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Timeline


- 2000: DPs for termination [Arts & Giesl 2000, . . . ]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"

- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $(\mathcal{DP}(\mathcal{R}), \mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Timeline


- 2000: DPs for termination [Arts & Giesl 2000, ...]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"

- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})$ / $(\mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Timeline


- 2000: DPs for termination [Arts & Giesl 2000, . . . ]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"

- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})$ / $(\mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Timeline




- 2000: DPs for termination [Arts & Giesl 2000, ...]
- 2006: Problem #106 of the RTA list of open problems:

  "*Can we use the dependency pair method to prove relative termination?*"

- 2016: Sufficient conditions on $\mathcal{R}$ and $\mathcal{B}$ such that:
  $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})$ / $(\mathcal{R} \cup \mathcal{B})$ is terminating [Iborra, Nishida, Vidal, & Yamada 2016]
- 2024: Annotated Dependency Pairs for Probabilistic Rewriting [Kassing & Giesl et al. 2024]
- Now: Annotated Dependency Pairs for Relative Termination

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $\mathsf{a} \rightarrow \mathsf{b}$ $\qquad$ $\mathcal{B}_1$ : $\mathsf{f} \rightarrow \mathsf{d}(\mathsf{a}, \mathsf{f})$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating:

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: f

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: f $\to_{\mathcal{B}_1}$ d(a, f)

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: f $\to_{\mathcal{B}_1}$ d(a, f) $\to_{\mathcal{R}_1}$ d(b, f)

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: f $\to_{\mathcal{B}_1}$ d(a, f) $\to_{\mathcal{R}_1}$ d(b, f) $\to_{\mathcal{B}_1}$ d(b, d(a, f))

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: f $\to_{\mathcal{B}_1}$ d(a, f) $\to_{\mathcal{R}_1}$ d(b, f) $\to_{\mathcal{B}_1}$ d(b, d(a, f)) $\to_{\mathcal{R}_1}$ ...

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{B}_1$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}, \mathsf{f})$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{B}_1$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}, \mathsf{f})$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{B}_2$ : $\mathsf{f}(x) \to \mathsf{d}(x, \mathsf{f}(x))$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : a $\to$ b $\qquad$ $\mathcal{B}_2$ : $\mathsf{f}(x) \to \mathsf{d}(x, \mathsf{f}(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating:

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $a \to b$ $\qquad$ $\mathcal{B}_1$ : $f \to d(a, f)$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a, f) \to_{\mathcal{R}_1} d(b, f) \to_{\mathcal{B}_1} d(b, d(a, f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : $a \to b$ $\qquad$ $\mathcal{B}_2$ : $f(x) \to d(x, f(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $f(a)$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $a \to b$ $\qquad$ $\mathcal{B}_1$ : $f \to d(a, f)$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a, f) \to_{\mathcal{R}_1} d(b, f) \to_{\mathcal{B}_1} d(b, d(a, f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : $a \to b$ $\qquad$ $\mathcal{B}_2$ : $f(x) \to d(x, f(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $f(a) \to_{\mathcal{B}_2} d(a, f(a))$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $a \to b$ $\qquad$ $\mathcal{B}_1$ : $f \to d(a, f)$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a, f) \to_{\mathcal{R}_1} d(b, f) \to_{\mathcal{B}_1} d(b, d(a, f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : $a \to b$ $\qquad$ $\mathcal{B}_2$ : $f(x) \to d(x, f(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $f(a) \to_{\mathcal{B}_2} d(a, f(a)) \to_{\mathcal{R}_2} d(b, f(a))$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : $a \to b$ $\qquad$ $\mathcal{B}_1$ : $f \to d(a, f)$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a, f) \to_{\mathcal{R}_1} d(b, f) \to_{\mathcal{B}_1} d(b, d(a, f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : $a \to b$ $\qquad$ $\mathcal{B}_2$ : $f(x) \to d(x, f(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $f(a) \to_{\mathcal{B}_2} d(a, f(a)) \to_{\mathcal{R}_2} d(b, f(a)) \to_{\mathcal{B}_2} d(b, d(a, f(a)))$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : a $\to$ b $\qquad$ $\mathcal{B}_2$ : f($x$) $\to$ d($x$, f($x$))

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $\mathsf{f}(\mathsf{a}) \to_{\mathcal{B}_2} \mathsf{d}(\mathsf{a},\mathsf{f}(\mathsf{a})) \to_{\mathcal{R}_2} \mathsf{d}(\mathsf{b},\mathsf{f}(\mathsf{a})) \to_{\mathcal{B}_2} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f}(\mathsf{a}))) \to_{\mathcal{R}_2} \ldots$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_2$ : a $\to$ b $\qquad$ $\mathcal{B}_2$ : $\mathsf{f}(x) \to \mathsf{d}(x, \mathsf{f}(x))$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $\mathsf{f}(\mathsf{a}) \to_{\mathcal{B}_2} \mathsf{d}(\mathsf{a},\mathsf{f}(\mathsf{a})) \to_{\mathcal{R}_2} \mathsf{d}(\mathsf{b},\mathsf{f}(\mathsf{a})) \to_{\mathcal{B}_2} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f}(\mathsf{a}))) \to_{\mathcal{R}_2} \ldots$

$\mathcal{DP}(\mathcal{R}_2)/(\mathcal{R}_2 \cup \mathcal{B}_2)$ is terminating: $\mathcal{DP}(\mathcal{R}_2) = \varnothing$

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$$\mathcal{R}_1: \quad a \to b \qquad\qquad \mathcal{B}_1: \quad f \to d(a, f)$$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a,f) \to_{\mathcal{R}_1} d(b,f) \to_{\mathcal{B}_1} d(b,d(a,f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$$\mathcal{R}_2: \quad a \to b \qquad\qquad \mathcal{B}_2: \quad f(x) \to d(x, f(x))$$

$\mathcal{R}_2/\mathcal{B}_2$ is not terminating: $f(a) \to_{\mathcal{B}_2} d(a,f(a)) \to_{\mathcal{R}_2} d(b,f(a)) \to_{\mathcal{B}_2} d(b,d(a,f(a))) \to_{\mathcal{R}_2} \ldots$

$\mathcal{DP}(\mathcal{R}_2)/(\mathcal{R}_2 \cup \mathcal{B}_2)$ is terminating: $\mathcal{DP}(\mathcal{R}_2) = \varnothing$

**Theorem**: DPs for Relative Termination [Iborra et al. 2016]

If $\mathcal{R}$ dominates $\mathcal{B}$ and $\mathcal{B}$ is non-duplicating, then $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$ is terminating

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$$\mathcal{R}_1: \quad a \to b \qquad\qquad \mathcal{B}_1: \quad f \to d(a, f)$$

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $f \to_{\mathcal{B}_1} d(a,f) \to_{\mathcal{R}_1} d(b,f) \to_{\mathcal{B}_1} d(b,d(a,f)) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$$\begin{aligned}
\mathcal{R}_{\text{divL}}: \quad \text{minus}(x,0) &\to x \\
\text{minus}(s(x),s(y)) &\to \text{minus}(x,y) \\
\text{div}(0,s(y)) &\to 0 \\
\text{div}(s(x),s(y)) &\to s(\text{div}(\text{minus}(x,y),s(y))) \\
\\
\text{divL}(x,\text{nil}) &\to x \\
\text{divL}(x,\text{cons}(y,xs)) &\to \text{divL}(\text{div}(x,y),xs) \\
\\
\mathcal{B}_{\text{com}}: \quad \text{divL}(z,\text{cons}(x,\text{cons}(y,xs))) &\to \text{divL}(z,\text{cons}(y,\text{cons}(x,xs)))
\end{aligned}$$

**Theorem**: DPs for Relative Termination [Iborra et al. 2016]

If $\mathcal{R}$ dominates $\mathcal{B}$ and $\mathcal{B}$ is non-duplicating, then $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$ is terminating

# Termination of $\mathcal{R}/\mathcal{B}$ and $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$

$\mathcal{R}_1$ : a $\to$ b $\qquad$ $\mathcal{B}_1$ : f $\to$ d(a, f)

$\mathcal{R}_1/\mathcal{B}_1$ is not terminating: $\mathsf{f} \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}_1} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}_1} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}_1} \ldots$

$\mathcal{DP}(\mathcal{R}_1)/(\mathcal{R}_1 \cup \mathcal{B}_1)$ is terminating: $\mathcal{DP}(\mathcal{R}_1) = \varnothing$

**Domination**

$\mathcal{R}$ dominates $\mathcal{B}$ iff no defined symbol of $\mathcal{R}$ is in a right-hand side of $\mathcal{B}$

$\mathcal{R}_{\mathsf{divL}}$ does not dominate $\mathcal{B}_{\mathsf{com}}$

$\mathcal{R}_{\mathsf{divL}}$ :

$$\begin{aligned}
\mathsf{minus}(x,0) &\to x\\
\mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}(x,y)\\
\mathsf{div}(0,\mathsf{s}(y)) &\to 0\\
\mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{div}(\mathsf{minus}(x,y),\mathsf{s}(y)))\\
\\
\mathsf{divL}(x,\mathsf{nil}) &\to x\\
\mathsf{divL}(x,\mathsf{cons}(y,xs)) &\to \mathsf{divL}(\mathsf{div}(x,y),xs)
\end{aligned}$$

$\mathcal{B}_{\mathsf{com}}$ :

$$\mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))$$

**Theorem**: DPs for Relative Termination [Iborra et al. 2016]

If $\mathcal{R}$ dominates $\mathcal{B}$ and $\mathcal{B}$ is non-duplicating, then $\mathcal{R}/\mathcal{B}$ is terminating iff $\mathcal{DP}(\mathcal{R})/(\mathcal{R} \cup \mathcal{B})$ is terminating

# Annotated Dependency Pairs

$\mathcal{R}$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

# Annotated Dependency Pairs

$\mathcal{R}$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$:

# Annotated Dependency Pairs

$\mathcal{R}$ : a $\to$ b $\qquad\qquad$ $\mathcal{B}$ : f $\to$ d(a ,f )

$\mathcal{R}/\mathcal{B}$ is not terminating: $\mathsf{f} \to_{\mathcal{B}} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$:
  - ADP for a $\to$ b needed although b is constructor

# Annotated Dependency Pairs

$\mathcal{R}$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
  - ADP for $a \to b$ needed although b is constructor

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{B}$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}, \mathsf{f})$

$\mathcal{R}/\mathcal{B}$ is not terminating: $\mathsf{f} \to_{\mathcal{B}} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
  - ADP for $\mathsf{a} \to \mathsf{b}$ needed although b is constructor

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$:

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$:
  - a is the $\mathcal{R}$-redex

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$:
  - a is the $\mathcal{R}$-redex
  - f creates infinitely many $\mathcal{R}$-redexes

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{B}$ : $f \to d(a, f)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible
  - a is the $\mathcal{R}$-redex
  - f creates infinitely many $\mathcal{R}$-redexes

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible
  - a is the $\mathcal{R}$-redex
  - f creates infinitely many $\mathcal{R}$-redexes

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

**Rewrite Relation for Set of ADPs $\mathcal{P}$**

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_\mathcal{B} d(a,f) \to_\mathcal{R} d(b,f) \to_\mathcal{B} d(b,d(a,f)) \to_\mathcal{R} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_\mathcal{P}$: rewrite step at #-symbol

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_\mathcal{B} d(a,f) \to_\mathcal{R} d(b,f) \to_\mathcal{B} d(b,d(a,f)) \to_\mathcal{R} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_\mathcal{P}$: rewrite step at #-symbol

- # in rhs of rules are kept

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $\mathsf{f} \to_{\mathcal{B}} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_\mathcal{P}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_\mathcal{P}$: rewrite step at non-#-symbol

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a, f) \to_{\mathcal{R}} d(b, f) \to_{\mathcal{B}} d(b, d(a, f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow_{\mathcal{P}}^{\#}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $\mathsf{f} \to_{\mathcal{B}} \mathsf{d}(\mathsf{a},\mathsf{f}) \to_{\mathcal{R}} \mathsf{d}(\mathsf{b},\mathsf{f}) \to_{\mathcal{B}} \mathsf{d}(\mathsf{b},\mathsf{d}(\mathsf{a},\mathsf{f})) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^{\#}_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a, f) \to_{\mathcal{R}} d(b, f) \to_{\mathcal{B}} d(b, d(a, f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow_{\mathcal{P}}^{\#}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

**Example**: $\mathcal{P} = \{f(x) \to g^\#(x)\}$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a, f) \to_{\mathcal{R}} d(b, f) \to_{\mathcal{B}} d(b, d(a, f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

**Example**: $\mathcal{P} = \{f(x) \to g^\#(x)\}$

$f^\#(a^\#) \quad \hookrightarrow^\#_{\mathcal{P}} \quad g^\#(a^\#)$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a, f) \to_{\mathcal{R}} d(b, f) \to_{\mathcal{B}} d(b, d(a, f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow_{\mathcal{P}}^{\#}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

**Example**: $\mathcal{P} = \{f(x) \to g^\#(x)\}$

$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a^\#)$
$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a)$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow_{\mathcal{P}}^{\#}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

**Example**: $\mathcal{P} = \{f(x) \to g^\#(x)\}$

$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a^\#)$
$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a)$

$f(a^\#) \hookrightarrow_{\mathcal{P}} g(a^\#)$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow_{\mathcal{P}}^{\#}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

**Example**: $\mathcal{P} = \{f(x) \to g^\#(x)\}$

$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a^\#)$ $\qquad$ $f(a^\#) \hookrightarrow_{\mathcal{P}} g(a^\#)$

$f^\#(a^\#) \hookrightarrow_{\mathcal{P}}^{\#} g^\#(a)$ $\qquad$ $f(a^\#) \hookrightarrow_{\mathcal{P}} g(a)$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P},\mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P},\mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \ldots$

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $\quad a \to b \qquad\qquad \mathcal{A}_2(\mathcal{B})$ : $\quad f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P},\mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P},\mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \ldots$

**Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

- ADPs for $\mathcal{R}$: 1 defined symbol in rhs annotated if possible
- ADPs for $\mathcal{B}$: 2 defined symbols in rhs annotated if possible

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P},\mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P},\mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \; \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; t_1 \; \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; \ldots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating:

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \ldots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating: $f^\#$

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \; \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; t_1 \; \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; \ldots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \dots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating: $f^\# \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(a^\#, f^\#)$

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \dots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating: $f^\# \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(a^\#, f^\#) \hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} d(b, f^\#)$

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \ldots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \ldots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating: $f^\# \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(a^\#, f^\#) \hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} d(b, f^\#) \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(b, d(a^\#, f^\#))$

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \ldots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Annotated Dependency Pairs

$\mathcal{A}_1(\mathcal{R})$ : $a \to b$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $f \to d(a^\#, f^\#)$

$\mathcal{R}/\mathcal{B}$ is not terminating: $f \to_{\mathcal{B}} d(a,f) \to_{\mathcal{R}} d(b,f) \to_{\mathcal{B}} d(b,d(a,f)) \to_{\mathcal{R}} \dots$

$(\mathcal{A}_1(\mathcal{R}), \mathcal{A}_2(\mathcal{B}))$ is not terminating: $f^\# \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(a^\#, f^\#) \hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} d(b, f^\#) \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} d(b, d(a^\#, f^\#)) \hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} \dots$

## Rewrite Relation for Set of ADPs $\mathcal{P}$

$\hookrightarrow^\#_{\mathcal{P}}$: rewrite step at #-symbol

- # in rhs of rules are kept
- # in variable positions can be kept

$\hookrightarrow_{\mathcal{P}}$: rewrite step at non-#-symbol

- # in rhs of rules are removed
- # in variable positions can be kept

## Termination of $(\mathcal{P}, \mathcal{S})$ for Sets of ADPs $\mathcal{P}$, $\mathcal{S}$

$(\mathcal{P}, \mathcal{S})$ is terminating iff there is no infinite evaluation

$t_0 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \; t_1 \hookrightarrow^\#_{\mathcal{P}} \circ (\hookrightarrow_{\mathcal{P}} \cup \hookrightarrow^\#_{\mathcal{S}} \cup \hookrightarrow_{\mathcal{S}})^* \dots$

## **Theorem**: Chain Criterion (if $\mathcal{B}$ is non-duplicating)

$\mathcal{R}/\mathcal{B}$ is terminating iff $\hookrightarrow^\#_{\mathcal{A}_1(\mathcal{R})} / (\hookrightarrow_{\mathcal{A}_1(\mathcal{R})} \cup \hookrightarrow^\#_{\mathcal{A}_2(\mathcal{B})} \cup \hookrightarrow_{\mathcal{A}_2(\mathcal{B})})$ is terminating

# Relative Dependency Pair Framework

- Objects that we work with:
  - (Relative) ADP Problems $(\mathcal{P}, \mathcal{S})$ with sets of ADPs $\mathcal{P}$, $\mathcal{S}$

# Relative Dependency Pair Framework

- Objects that we work with:
  - (Relative) ADP Problems $(\mathcal{P}, \mathcal{S})$ with sets of ADPs $\mathcal{P}$, $\mathcal{S}$
- Initial DP problem for $\mathcal{R}/\mathcal{B}$:
  - (Chain Criterion) Use all main ADPs with 1-annotation $\mathcal{A}_1(\mathcal{R})$ and all base ADPs with 2-annotations $\mathcal{A}_2(\mathcal{B})$

# Relative Dependency Pair Framework

- Objects that we work with:
  - (Relative) ADP Problems $(\mathcal{P}, \mathcal{S})$ with sets of ADPs $\mathcal{P}$, $\mathcal{S}$
- Initial DP problem for $\mathcal{R}/\mathcal{B}$:
  - (Chain Criterion) Use all main ADPs with 1-annotation $\mathcal{A}_1(\mathcal{R})$ and all base ADPs with 2-annotations $\mathcal{A}_2(\mathcal{B})$
- How do we create smaller problems?:
  - DP Processors: $\mathit{Proc}(\mathcal{P}, \mathcal{S}) = \{(\mathcal{P}_1, \mathcal{S}_1), \ldots, (\mathcal{P}_k, \mathcal{S}_k)\}$

# Relative Dependency Pair Framework

- Objects that we work with:
  - (Relative) ADP Problems $(\mathcal{P}, \mathcal{S})$ with sets of ADPs $\mathcal{P}$, $\mathcal{S}$
- Initial DP problem for $\mathcal{R}/\mathcal{B}$:
  - (Chain Criterion) Use all main ADPs with 1-annotation $\mathcal{A}_1(\mathcal{R})$ and all base ADPs with 2-annotations $\mathcal{A}_2(\mathcal{B})$
- How do we create smaller problems?:
  - DP Processors: $\mathit{Proc}(\mathcal{P}, \mathcal{S}) = \{(\mathcal{P}_1, \mathcal{S}_1), \ldots, (\mathcal{P}_k, \mathcal{S}_k)\}$
  - *Proc* is sound: if all $(\mathcal{P}_i, \mathcal{S}_i)$ are terminating, then $(\mathcal{P}, \mathcal{S})$ is terminating

# Relative Dependency Pair Framework

- Objects that we work with:
  - (Relative) ADP Problems $(\mathcal{P}, \mathcal{S})$ with sets of ADPs $\mathcal{P}$, $\mathcal{S}$
- Initial DP problem for $\mathcal{R}/\mathcal{B}$:
  - (Chain Criterion) Use all main ADPs with 1-annotation $\mathcal{A}_1(\mathcal{R})$ and all base ADPs with 2-annotations $\mathcal{A}_2(\mathcal{B})$
- How do we create smaller problems?:
  - DP Processors: $\mathit{Proc}(\mathcal{P}, \mathcal{S}) = \{(\mathcal{P}_1, \mathcal{S}_1), \ldots, (\mathcal{P}_k, \mathcal{S}_k)\}$
  - *Proc* is sound: if all $(\mathcal{P}_i, \mathcal{S}_i)$ are terminating, then $(\mathcal{P}, \mathcal{S})$ is terminating
  - *Proc* is complete: if $(\mathcal{P}, \mathcal{S})$ is terminating, then all $(\mathcal{P}_i, \mathcal{S}_i)$ are terminating

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\quad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^{\#}, \mathsf{f}^{\#})$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\quad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^{\#}, \mathsf{f}^{\#})$

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

# Dependency Graph Processor

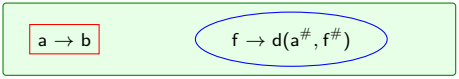
$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\quad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^{\#}, \mathsf{f}^{\#})$

**$(\mathcal{P}, \mathcal{S})$-Dependency Graph**

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\quad a \to b$ $\qquad\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\quad f \to d(a^\#, f^\#)$

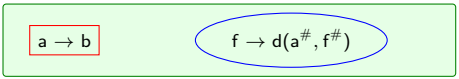



## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$



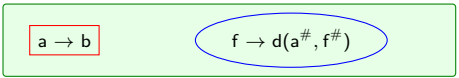

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\mathsf{b}(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

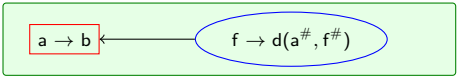
only $\mathrm{root}(t)$ annotated

delete all annotations in $\mathcal{P} \cup \mathcal{S}$

only $\mathrm{root}(\ell_2)$ annotated

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$



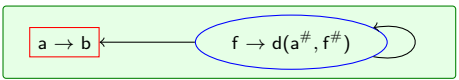

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\mathsf{b}(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$



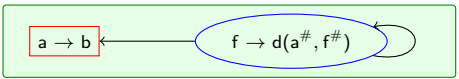

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\mathsf{b}(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

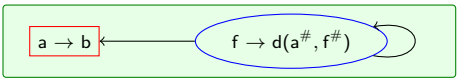
$$t^\# \sigma_1 \to^*_{\mathsf{b}(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

## **Dependency Graph Processor** (sound & complete)

$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{$ $\qquad$ $|$ $\qquad$ $\}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\mathsf{b}(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

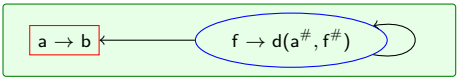
## **Dependency Graph Processor** (sound & complete)

$$\mathit{Proc}_{DG}(\mathcal{P}, \mathcal{S}) = \{ \quad \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \quad \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\quad a \to b \qquad\qquad \mathcal{A}_2(\mathcal{B})$ : $\quad f \to d(a^\#, f^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

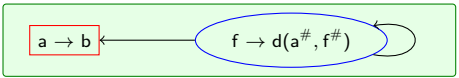
## **Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\ \mathcal{P} \cap \mathcal{Q},\ \ \mathcal{S} \cap \mathcal{Q}\ ) \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\quad a \to b$ $\qquad\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\quad f \to d(a^\#, f^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

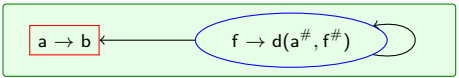
## **Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ ( \mathcal{P} \cap \mathcal{Q}, (\mathcal{S} \cap \mathcal{Q}) \cup \flat( (\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q} ) ) \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

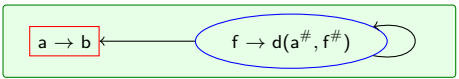
## **Dependency Graph Processor** (sound & complete)

$$\mathit{Proc}_{DG}(\mathcal{P}, \mathcal{S}) \;=\; \{\, (\, \mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat(\, (\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q}\, )\, ) \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \cup \mathtt{Lasso} \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




$Proc_{DG}(\mathcal{A}_1(\mathcal{R}),\ \mathcal{A}_2(\mathcal{B})) = (\mathcal{A}_1(\mathcal{R}),\ \mathcal{A}_2(\mathcal{B}))$

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

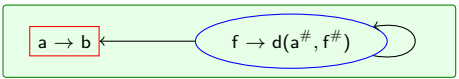
## **Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat((\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q})) \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \cup \mathtt{Lasso} \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R})$ : $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathcal{A}_2(\mathcal{B})$ : $\mathsf{f} \to \mathsf{d}(\mathsf{a}^\#, \mathsf{f}^\#)$




$Proc_{DG}(\mathcal{A}_1(\mathcal{R}),\ \mathcal{A}_2(\mathcal{B})) = (\mathcal{A}_1(\mathcal{R}),\ \mathcal{A}_2(\mathcal{B}))$

## $(\mathcal{P}, \mathcal{S})$-Dependency Graph

- nodes: ADPs from $\mathcal{P} \cup \mathcal{S}$
- edge from $\ell_1 \to r_1$ to $\ell_2 \to r_2$ if there is a subterm $t$ of $r_1$ at annotated position such that

$$t^\# \sigma_1 \to^*_{\flat(\mathcal{P} \cup \mathcal{S})} \ell_2^\# \sigma_2$$

## **Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat((\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q})) \mid \mathcal{Q} \in \mathtt{SCC}_\mathcal{P} \cup \mathtt{Lasso} \}$$

- $\mathtt{SCC}_\mathcal{P}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

} removes all annotations from $\mathcal{S}$ if $\mathcal{P}$ dominates $\mathcal{S}$
$\Rightarrow$ subsumes Thm. of [Iborra et al. 2016]

# Dependency Graph Processor

$$
\begin{array}{lllll}
\mathcal{R}_{\mathsf{divL}} : & (a) & \mathsf{minus}(x, 0) & \to x \\
& (b) & \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to \mathsf{minus}\ \ (x, y) \\
& (c) & \mathsf{div}(0, \mathsf{s}(y)) & \to 0 \\
& (d) & \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}(\mathsf{minus}\ \ (x, y), \mathsf{s}(y))) \\
& (e) & \mathsf{divL}(x, \mathsf{nil}) & \to x \\
& (f) & \mathsf{divL}(x, \mathsf{cons}(y, xs)) & \to \mathsf{divL}(\mathsf{div}\ \ (x, y), xs) \\
\mathcal{B}_{\mathsf{com}} : & (g) & \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) & \to \mathsf{divL}\ \ (z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))
\end{array}
$$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\mathsf{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{minus}\ \ (x, y)$ |
| $(c)$ | $\mathsf{div}(0, \mathsf{s}(y))$ | $\to 0$ |
| $(d)$ | $\mathsf{div}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{s}(\mathsf{div}(\mathsf{minus}\ \ (x, y), \mathsf{s}(y)))$ |
| $(e)$ | $\mathsf{divL}(x, \mathsf{nil})$ | $\to x$ |
| $(f)$ | $\mathsf{divL}(x, \mathsf{cons}(y, xs))$ | $\to \mathsf{divL}(\mathsf{div}\ \ (x, y), xs)$ |

$\mathcal{B}_{\mathsf{com}}$ :

| | | |
|---|---|---|
| $(g)$ | $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs)))$ | $\to \mathsf{divL}\ \ (z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$ |

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\mathsf{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{minus}^{\#}(x, y)$ |
| $(c)$ | $\mathsf{div}(0, \mathsf{s}(y))$ | $\to 0$ |
| $(d)$ | $\mathsf{div}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{s}(\mathsf{div}(\mathsf{minus}\ \ (x, y), \mathsf{s}(y)))$ |
| $(e)$ | $\mathsf{divL}(x, \mathsf{nil})$ | $\to x$ |
| $(f)$ | $\mathsf{divL}(x, \mathsf{cons}(y, xs))$ | $\to \mathsf{divL}(\mathsf{div}\ \ (x, y), xs)$ |

$\mathcal{B}_{\mathsf{com}}$ :

| | | |
|---|---|---|
| $(g)$ | $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs)))$ | $\to \mathsf{divL}\ \ (z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$ |

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\mathsf{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{minus}^{\#}(x, y)$ |
| $(c)$ | $\mathsf{div}(0, \mathsf{s}(y))$ | $\to 0$ |
| $(d1)$ | $\mathsf{div}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x, y), \mathsf{s}(y)))$ |
| $(e)$ | $\mathsf{divL}(x, \mathsf{nil})$ | $\to x$ |
| $(f)$ | $\mathsf{divL}(x, \mathsf{cons}(y, xs))$ | $\to \mathsf{divL}(\mathsf{div} \quad (x, y), xs)$ |

$\mathcal{B}_{\mathsf{com}}$ :

| | | |
|---|---|---|
| $(g)$ | $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs)))$ | $\to \mathsf{divL} \quad (z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$ |

# Dependency Graph Processor

$$
\begin{array}{lll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) & \mathsf{minus}(x,0) \to x \\
& (b) & \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^{\#}(x,y) \\
& (c) & \mathsf{div}(0,\mathsf{s}(y)) \to 0 \\
& (d1) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
& (d2) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
& (e) & \mathsf{divL}(x,\mathsf{nil}) \to x \\
& (f) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) \to \mathsf{divL}(\mathsf{div}\ \ (x,y),xs) \\
\\
\mathcal{B}_{\mathsf{com}}: & (g) & \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}\ \ (z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$

# Dependency Graph Processor

$$
\begin{array}{lrll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) & \mathsf{minus}(x,0) & \to x \\
& (b) & \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{minus}^{\#}(x,y) \\
& (c) & \mathsf{div}(0,\mathsf{s}(y)) & \to 0 \\
& (d1) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
& (d2) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
& (e) & \mathsf{divL}(x,\mathsf{nil}) & \to x \\
& (f1) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}(\mathsf{div}^{\#}(x,y),xs) \\
\\
\mathcal{B}_{\mathsf{com}}: & (g) & \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) & \to \mathsf{divL}\quad(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$

# Dependency Graph Processor

$$
\begin{array}{rlll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) & \mathsf{minus}(x,0) & \to x \\
 & (b) & \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{minus}^{\#}(x,y) \\
 & (c) & \mathsf{div}(0,\mathsf{s}(y)) & \to 0 \\
 & (d1) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
 & (d2) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
 & (e) & \mathsf{divL}(x,\mathsf{nil}) & \to x \\
 & (f1) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}(\mathsf{div}^{\#}(x,y),xs) \\
 & (f2) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs) \\
\mathcal{B}_{\mathsf{com}}: & (g) & \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) & \to \mathsf{divL}\quad(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$

# Dependency Graph Processor

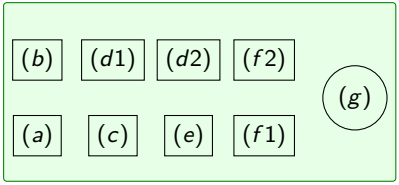
$$
\begin{array}{lll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) \quad \mathsf{minus}(x,0) & \to x \\
& (b) \quad \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{minus}^{\#}(x,y) \\
& (c) \quad \mathsf{div}(0,\mathsf{s}(y)) & \to 0 \\
& (d1) \quad \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
& (d2) \quad \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
& (e) \quad \mathsf{divL}(x,\mathsf{nil}) & \to x \\
& (f1) \quad \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}(\mathsf{div}^{\#}(x,y),xs) \\
& (f2) \quad \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs) \\
\mathcal{A}_2(\mathcal{B}_{\mathsf{com}}): & (g) \quad \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) & \to \mathsf{divL} \quad (z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$

# Dependency Graph Processor

$$
\begin{array}{lll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) \quad \mathsf{minus}(x,0) & \to x \\
& (b) \quad \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{minus}^{\#}(x,y) \\
& (c) \quad \mathsf{div}(0,\mathsf{s}(y)) & \to 0 \\
& (d1) \quad \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
& (d2) \quad \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) & \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
& (e) \quad \mathsf{divL}(x,\mathsf{nil}) & \to x \\
& (f1) \quad \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}(\mathsf{div}^{\#}(x,y),xs) \\
& (f2) \quad \mathsf{divL}(x,\mathsf{cons}(y,xs)) & \to \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs) \\
\mathcal{A}_2(\mathcal{B}_{\mathsf{com}}): & (g) \quad \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) & \to \mathsf{divL}^{\#}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$

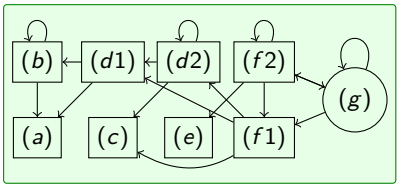
# Dependency Graph Processor

| | | | |
|---|---|---|---|
| $\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}})$ : | $(a)$ | $\mathsf{minus}(x, 0)$ | $\to x$ |
| | $(b)$ | $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{minus}^{\#}(x, y)$ |
| | $(c)$ | $\mathsf{div}(0, \mathsf{s}(y))$ | $\to 0$ |
| | $(d1)$ | $\mathsf{div}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x, y), \mathsf{s}(y)))$ |
| | $(d2)$ | $\mathsf{div}(\mathsf{s}(x), \mathsf{s}(y))$ | $\to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x, y), \mathsf{s}(y)))$ |
| | $(e)$ | $\mathsf{divL}(x, \mathsf{nil})$ | $\to x$ |
| | $(f1)$ | $\mathsf{divL}(x, \mathsf{cons}(y, xs))$ | $\to \mathsf{divL}(\mathsf{div}^{\#}(x, y), xs)$ |
| | $(f2)$ | $\mathsf{divL}(x, \mathsf{cons}(y, xs))$ | $\to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ |
| $\mathcal{A}_2(\mathcal{B}_{\mathsf{com}})$ : | $(g)$ | $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs)))$ | $\to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$ |



# Dependency Graph Processor

$$
\begin{array}{lll}
\mathcal{A}_1(\mathcal{R}_{\mathsf{divL}}): & (a) & \mathsf{minus}(x,0) \to x \\
& (b) & \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^{\#}(x,y) \\
& (c) & \mathsf{div}(0,\mathsf{s}(y)) \to 0 \\
& (d1) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{div}(\mathsf{minus}^{\#}(x,y),\mathsf{s}(y))) \\
& (d2) & \mathsf{div}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{div}^{\#}(\mathsf{minus}(x,y),\mathsf{s}(y))) \\
& (e) & \mathsf{divL}(x,\mathsf{nil}) \to x \\
& (f1) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) \to \mathsf{divL}(\mathsf{div}^{\#}(x,y),xs) \\
& (f2) & \mathsf{divL}(x,\mathsf{cons}(y,xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x,y),xs) \\
\mathcal{A}_2(\mathcal{B}_{\mathsf{com}}): & (g) & \mathsf{divL}(z,\mathsf{cons}(x,\mathsf{cons}(y,xs))) \to \mathsf{divL}^{\#}(z,\mathsf{cons}(y,\mathsf{cons}(x,xs)))
\end{array}
$$



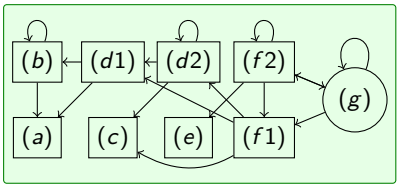
# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\text{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\text{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\text{minus}(\text{s}(x), \text{s}(y))$ | $\to \text{minus}^{\#}(x, y)$ |
| $(c)$ | $\text{div}(0, \text{s}(y))$ | $\to 0$ |
| $(d1)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}(\text{minus}^{\#}(x, y), \text{s}(y)))$ |
| $(d2)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}^{\#}(\text{minus}(x, y), \text{s}(y)))$ |
| $(e)$ | $\text{divL}(x, \text{nil})$ | $\to x$ |
| $(f1)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}(\text{div}^{\#}(x, y), xs)$ |
| $(f2)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}^{\#}(\text{div}(x, y), xs)$ |

$\mathcal{A}_2(\mathcal{B}_{\text{com}})$ :

| | | |
|---|---|---|
| $(g)$ | $\text{divL}(z, \text{cons}(x, \text{cons}(y, xs)))$ | $\to \text{divL}^{\#}(z, \text{cons}(y, \text{cons}(x, xs)))$ |




**Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat((\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q})) \mid \mathcal{Q} \in \mathtt{SCC}_{\mathcal{P}} \cup \mathtt{Lasso} \}$$

- $\mathtt{SCC}_{\mathcal{P}}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

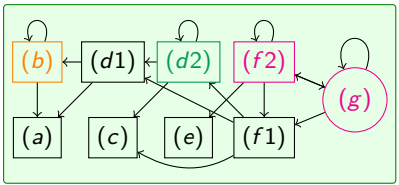
# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\text{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\text{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\text{minus}(\text{s}(x), \text{s}(y))$ | $\to \text{minus}^{\#}(x, y)$ |
| $(c)$ | $\text{div}(0, \text{s}(y))$ | $\to 0$ |
| $(d1)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}(\text{minus}^{\#}(x, y), \text{s}(y)))$ |
| $(d2)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}^{\#}(\text{minus}(x, y), \text{s}(y)))$ |
| $(e)$ | $\text{divL}(x, \text{nil})$ | $\to x$ |
| $(f1)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}(\text{div}^{\#}(x, y), xs)$ |
| $(f2)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}^{\#}(\text{div}(x, y), xs)$ |

$\mathcal{A}_2(\mathcal{B}_{\text{com}})$ :

| | | |
|---|---|---|
| $(g)$ | $\text{divL}(z, \text{cons}(x, \text{cons}(y, xs)))$ | $\to \text{divL}^{\#}(z, \text{cons}(y, \text{cons}(x, xs)))$ |




SCC: $\{(b)\}$, $\{(d2)\}$, and $\{(g), (f2)\}$ Lasso: none

**Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat((\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q})) \mid \mathcal{Q} \in \text{SCC}_{\mathcal{P}} \cup \text{Lasso} \}$$

- $\text{SCC}_{\mathcal{P}}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

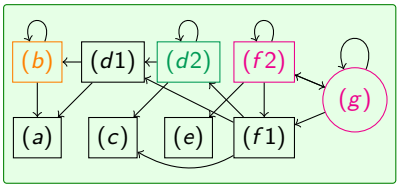
# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\text{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\text{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\text{minus}(\text{s}(x), \text{s}(y))$ | $\to \text{minus}^{\#}(x, y)$ |
| $(c)$ | $\text{div}(0, \text{s}(y))$ | $\to 0$ |
| $(d1)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}(\text{minus}\ \ (x, y), \text{s}(y)))$ |
| $(d2)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}^{\#}(\text{minus}(x, y), \text{s}(y)))$ |
| $(e)$ | $\text{divL}(x, \text{nil})$ | $\to x$ |
| $(f1)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}(\text{div}\ \ (x, y), xs)$ |
| $(f2)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}\ \ (\text{div}(x, y), xs)$ |

$\mathcal{A}_2(\mathcal{B}_{\text{com}})$ :

| | | |
|---|---|---|
| $(g)$ | $\text{divL}(z, \text{cons}(x, \text{cons}(y, xs)))$ | $\to \text{divL}\ \ (z, \text{cons}(y, \text{cons}(x, xs)))$ |




SCC: $\{(b)\}$, $\{(d2)\}$, and $\{(g), (f2)\}$ Lasso: none

**Derelatifying Processor** (sound & complete)

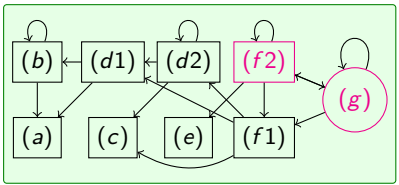
$$Proc_{DR}(\mathcal{P}, \mathcal{S}) = \varnothing$$

if $\mathcal{S}$ has no annotations and ordinary DP problem $(\text{dp}(\mathcal{P}), \flat(\mathcal{P} \cup \mathcal{S}))$ is terminating
where $\text{dp}(\mathcal{P}) = \{\, \ell^{\#} \to t^{\#} \mid \ell \to \ldots t^{\#} \ldots \in \mathcal{P} \,\}$

# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\text{divL}})$ :

$$
\begin{array}{lll}
(a) & \text{minus}(x,0) & \to x \\
(b) & \text{minus}(\text{s}(x),\text{s}(y)) & \to \text{minus}^{\#}(x,y) \\
(c) & \text{div}(0,\text{s}(y)) & \to 0 \\
(d1) & \text{div}(\text{s}(x),\text{s}(y)) & \to \text{s}(\text{div}(\text{minus}^{\#}(x,y),\text{s}(y))) \\
(d2) & \text{div}(\text{s}(x),\text{s}(y)) & \to \text{s}(\text{div}^{\#}(\text{minus}(x,y),\text{s}(y))) \\
(e) & \text{divL}(x,\text{nil}) & \to x \\
(f1) & \text{divL}(x,\text{cons}(y,xs)) & \to \text{divL}(\text{div}^{\#}(x,y),xs) \\
(f2) & \text{divL}(x,\text{cons}(y,xs)) & \to \text{divL}^{\#}(\text{div}(x,y),xs) \\
\mathcal{A}_2(\mathcal{B}_{\text{com}}) : \ (g) & \text{divL}(z,\text{cons}(x,\text{cons}(y,xs))) & \to \text{divL}^{\#}(z,\text{cons}(y,\text{cons}(x,xs)))
\end{array}
$$




SCC: $\{(b)\}$, $\{(d2)\}$, and $\{(g),(f2)\}$ Lasso: none

**Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P},\mathcal{S}) = \{ (\mathcal{P}\cap\mathcal{Q},\ (\mathcal{S}\cap\mathcal{Q}) \cup \flat((\mathcal{P}\cup\mathcal{S})\setminus\mathcal{Q})) \mid \mathcal{Q} \in \text{SCC}_{\mathcal{P}} \cup \text{Lasso}\}$$

- $\text{SCC}_{\mathcal{P}}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

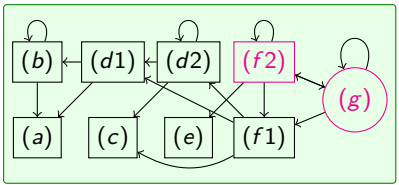
# Dependency Graph Processor

$\mathcal{A}_1(\mathcal{R}_{\text{divL}})$ :

| | | |
|---|---|---|
| $(a)$ | $\text{minus}(x, 0)$ | $\to x$ |
| $(b)$ | $\text{minus}(\text{s}(x), \text{s}(y))$ | $\to \text{minus}\ \ (x, y)$ |
| $(c)$ | $\text{div}(0, \text{s}(y))$ | $\to 0$ |
| $(d1)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}(\text{minus}\ \ (x, y), \text{s}(y)))$ |
| $(d2)$ | $\text{div}(\text{s}(x), \text{s}(y))$ | $\to \text{s}(\text{div}\ \ (\text{minus}(x, y), \text{s}(y)))$ |
| $(e)$ | $\text{divL}(x, \text{nil})$ | $\to x$ |
| $(f1)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}(\text{div}\ \ (x, y), xs)$ |
| $(f2)$ | $\text{divL}(x, \text{cons}(y, xs))$ | $\to \text{divL}^{\#}(\text{div}(x, y), xs)$ |

$\mathcal{A}_2(\mathcal{B}_{\text{com}})$ :

| | | |
|---|---|---|
| $(g)$ | $\text{divL}(z, \text{cons}(x, \text{cons}(y, xs)))$ | $\to \text{divL}^{\#}(z, \text{cons}(y, \text{cons}(x, xs)))$ |




SCC: $\{(b)\}$, $\{(d2)\}$, and $\{(g), (f2)\}$ Lasso: none

**Dependency Graph Processor** (sound & complete)

$$Proc_{DG}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \cap \mathcal{Q},\ (\mathcal{S} \cap \mathcal{Q}) \cup \flat((\mathcal{P} \cup \mathcal{S}) \setminus \mathcal{Q})) \mid \mathcal{Q} \in \text{SCC}_{\mathcal{P}} \cup \text{Lasso} \}$$

- $\text{SCC}_{\mathcal{P}}$: SCC with ADP from $\mathcal{P}$
- Lasso: SCC of $\mathcal{S}$ with 2 annotations, plus outgoing path to leaf from $\mathcal{P}$

# Reduction Pair Processor

$(f2)$ $\mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

$(g)$ $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

$(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\ \mathcal{P} \setminus \mathcal{P}_{>_{Pol}},\ \ \mathcal{S} \setminus \mathcal{P}_{>_{Pol}} \qquad )\} \quad \textit{if}$$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

$(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\ \mathcal{P} \setminus \mathcal{P}_{>Pol},\ (\mathcal{S} \setminus \mathcal{P}_{>Pol}) \cup \flat(\ \mathcal{P}_{>Pol}\ )\ )\} \quad \textit{if}$$

# Reduction Pair Processor

$(f2)$ $\mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

$(g)$ $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat(\mathcal{P}_{>_{Pol}})) \} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

$(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\ \mathcal{P} \setminus \mathcal{P}_{>_{Pol}},\ (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat(\ \mathcal{P}_{>_{Pol}}\ )\ )\} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)$ $\mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)$ $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation $Pol$:

$$\begin{array}{lcl} \mathsf{divL}^{\#}_{Pol}(x, xs) & = & xs \\ \mathsf{cons}_{Pol}(x, xs) & = & xs + 1 \\ & \ldots & \end{array}$$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat(\mathcal{P}_{>_{Pol}})) \} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)$ $\mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)$ $\mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation $Pol$:

$$\begin{aligned} \mathsf{divL}^{\#}_{Pol}(x, xs) &= xs \\ \mathsf{cons}_{Pol}(x, xs) &= xs + 1 \\ &\dots \end{aligned}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ ( \mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat( \mathcal{P}_{>_{Pol}} ) )\} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \dots t^{\#} \dots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\qquad$ $(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation $Pol$:

$$\begin{aligned} \mathsf{divL}^{\#}_{Pol}(x, xs) &= xs \\ \mathsf{cons}_{Pol}(x, xs) &= xs + 1 \\ &\dots \end{aligned}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$ and $\quad \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) \quad >_{Pol} \quad \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$

## **Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ (\mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat(\mathcal{P}_{>_{Pol}})) \} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \dots t^{\#} \dots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation *Pol*:

$$\begin{array}{lcl} \mathsf{divL}^{\#}_{Pol}(x, xs) & = & xs \\ \mathsf{cons}_{Pol}(x, xs) & = & xs + 1 \\ & \ldots & \end{array}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$ and

$$\mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) >_{Pol} \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$$

$$\mathsf{divL}^{\#}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \geq_{Pol} \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$$

**Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ ( \mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat( \mathcal{P}_{>_{Pol}} ) ) \} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation *Pol*:

$$\begin{array}{lcl} \mathsf{divL}^{\#}_{Pol}(x, xs) & = & xs \\ \mathsf{cons}_{Pol}(x, xs) & = & xs + 1 \\ & \ldots & \end{array}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$ and

$$\begin{array}{lcl} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) & >_{Pol} & \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \\ xs + 1 & > & xs \\ \mathsf{divL}^{\#}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) & \geq_{Pol} & \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs))) \end{array}$$

## **Reduction Pair Processor** (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ ( \mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat( \mathcal{P}_{>_{Pol}} ) )\} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation $Pol$:

$$\begin{array}{lcl} \mathsf{divL}^{\#}_{Pol}(x, xs) & = & xs \\ \mathsf{cons}_{Pol}(x, xs) & = & xs + 1 \\ & \ldots & \end{array}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$ and

$$\begin{array}{lcl} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) & >_{Pol} & \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \\ xs + 1 & > & xs \\ \mathsf{divL}^{\#}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) & \geq_{Pol} & \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs))) \\ xs + 2 & \geq & xs + 2 \end{array}$$

## Reduction Pair Processor (sound & complete)

$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{ ( \mathcal{P} \setminus \mathcal{P}_{>_{Pol}}, (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat( \mathcal{P}_{>_{Pol}} ) )\} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Reduction Pair Processor

$(f2)\ \mathsf{divL}(x, \mathsf{cons}(y, xs)) \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs)$ $\quad$ $(g)\ \mathsf{divL}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) \to \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs)))$

Find natural polynomial interpretation $Pol$:

$$\begin{array}{lcl} \mathsf{divL}^{\#}_{Pol}(x, xs) & = & xs \\ \mathsf{cons}_{Pol}(x, xs) & = & xs + 1 \\ & \ldots & \end{array}$$

Then: $\flat(\mathcal{P} \cup \mathcal{S}) \subseteq \geq_{Pol}$ and

$$\begin{array}{lcl} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) & >_{Pol} & \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \\ xs + 1 & > & xs \\ \mathsf{divL}^{\#}(z, \mathsf{cons}(x, \mathsf{cons}(y, xs))) & \geq_{Pol} & \mathsf{divL}^{\#}(z, \mathsf{cons}(y, \mathsf{cons}(x, xs))) \\ xs + 2 & \geq & xs + 2 \end{array}$$

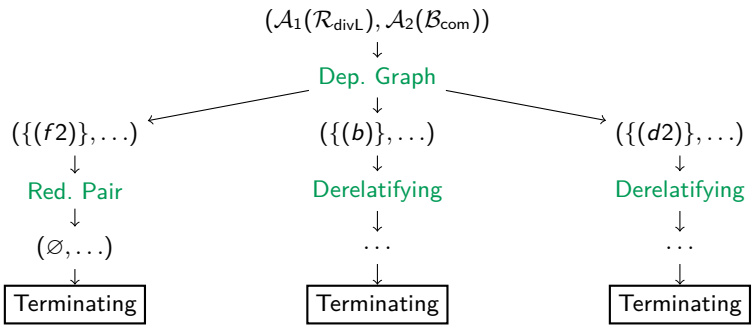
Thus: $Proc_{RP}(\ \{(f2)\},\ \ldots\ ) = \{(\varnothing, \ldots)\}$

**Reduction Pair Processor** (sound & complete)

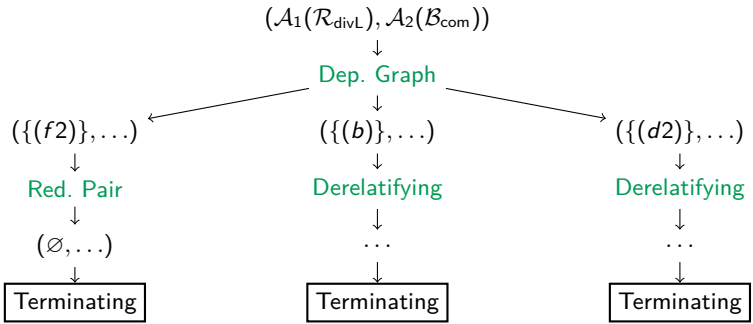
$$Proc_{RP}(\mathcal{P}, \mathcal{S}) = \{\ (\ \mathcal{P} \setminus \mathcal{P}_{>_{Pol}},\ (\mathcal{S} \setminus \mathcal{P}_{>_{Pol}}) \cup \flat(\ \mathcal{P}_{>_{Pol}}\ )\ )\} \quad \textit{if}$$

- $\ell^{\#} >_{Pol} t^{\#}$ for all $\ell \to \ldots t^{\#} \ldots \in \mathcal{P}_{>_{Pol}} \subseteq \mathcal{P} \cup \mathcal{S}$
- $\ell \geq_{Pol} r$ for all $\ell \to r \in \flat(\mathcal{P} \cup \mathcal{S})$

# Relative Termination Proof with ADPs



# Relative Termination Proof with ADPs




$\Rightarrow$ **Relative termination is proved automatically!**

# Implementation in AProVE

# Implementation in AProVE

**Relative rewriting** (130 benchmarks):

| | *new* AProVE | NaTT | *old* AProVE | $T_TT_2$ | MultumNonMulta |
|---|---|---|---|---|---|
| YES | 91 | 68 | 48 | 39 | 0 |
| NO | 13 | 5 | 13 | 7 | 13 |

# Implementation in AProVE

**Relative rewriting** (130 benchmarks):

| | *new* AProVE | NaTT | *old* AProVE | $T_TT_2$ | MultumNonMulta |
|---|---|---|---|---|---|
| YES | 91 | 68 | 48 | 39 | 0 |
| NO | 13 | 5 | 13 | 7 | 13 |

**Relative string rewriting** (403 benchmarks):

| | MultumNonMulta | Matchbox | AProVE | ADPs |
|---|---|---|---|---|
| YES | 261 | 259 | 207 | 71 |

# Implementation in AProVE

**Relative rewriting** (130 benchmarks):

| | *new* AProVE | NaTT | *old* AProVE | $T_TT_2$ | MultumNonMulta |
|---|---|---|---|---|---|
| YES | 91 | 68 | 48 | 39 | 0 |
| NO | 13 | 5 | 13 | 7 | 13 |

**Relative string rewriting** (403 benchmarks):

| | MultumNonMulta | Matchbox | AProVE | ADPs |
|---|---|---|---|---|
| YES | 261 | 259 | 207 | 71 |

**Equational rewriting** (76 benchmarks):

| | AProVE | MU-TERM | ADPs |
|---|---|---|---|
| YES | 66 | 64 | 36 |

# Conclusion

- First specific DP framework for relative termination

# Conclusion

- First specific DP framework for relative termination
- Annotated Dependency Pairs:

$$\begin{array}{ll} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) & \to \mathsf{divL}(\mathsf{div}^{\#}(x, y), xs) \\ \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) & \to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \end{array}$$

# Conclusion

- First specific DP framework for relative termination
- Annotated Dependency Pairs:

$$\begin{aligned} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}(\mathsf{div}^{\#}(x, y), xs) \\ \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \end{aligned}$$

- Adapted the core DP processors:
  - Dependency Graph Processor
  - Derelatifying Processor
  - Reduction Pair Processor
  - Rule Removal Processor

# Conclusion

- First specific DP framework for relative termination
- Annotated Dependency Pairs:

$$\begin{aligned} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}(\mathsf{div}^{\#}(x, y), xs) \\ \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \end{aligned}$$

- Adapted the core DP processors:
  - Dependency Graph Processor
  - Derelatifying Processor
  - Reduction Pair Processor
  - Rule Removal Processor
- Implemented in AProVE

# Conclusion

- First specific DP framework for relative termination
- Annotated Dependency Pairs:

$$\begin{aligned} \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}(\mathsf{div}^{\#}(x, y), xs) \\ \mathsf{divL}^{\#}(x, \mathsf{cons}(y, xs)) &\to \mathsf{divL}^{\#}(\mathsf{div}(x, y), xs) \end{aligned}$$

- Adapted the core DP processors:
  - Dependency Graph Processor
  - Derelatifying Processor
  - Reduction Pair Processor
  - Rule Removal Processor
- Implemented in AProVE
- Future work:
  - Further processors to (dis)-prove relative termination
  - ADPs to analyze further properties