Lower Bounds for Runtime Complexity of Term Rewriting

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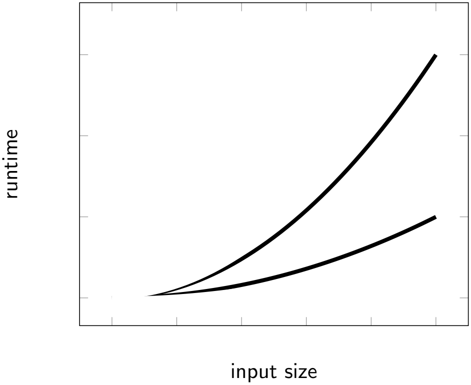
joint work with Florian Frohn, Jera Hensel, Cornelius Aschermann, and Thomas Ströder

Lower Bounds?



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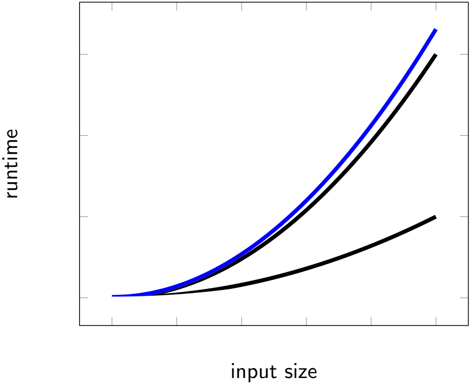
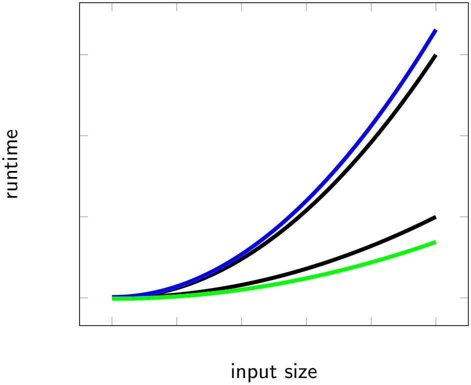


- worst-case upper bounds

Lower Bounds?



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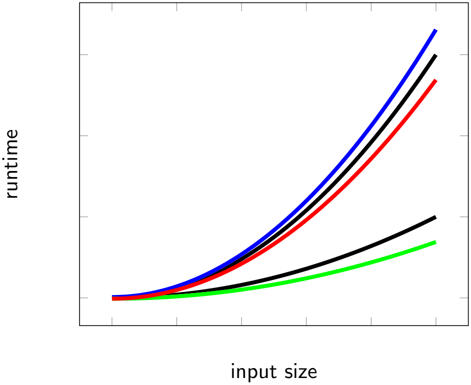

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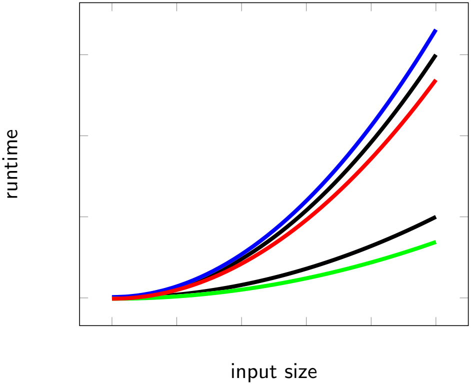

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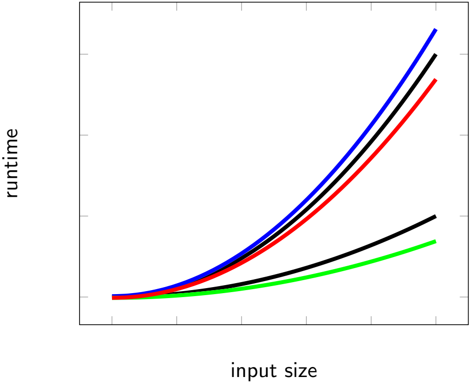
Why?

- *tight* bounds

Lower Bounds?



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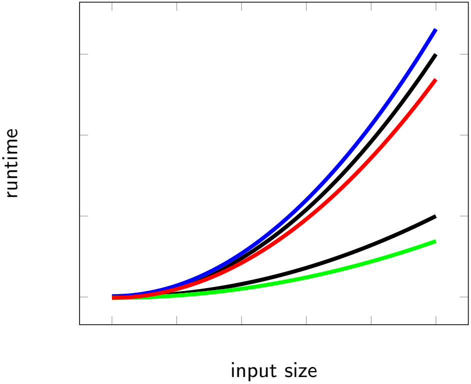

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Why?

- *tight* bounds
- detect bugs
- detect potential attacks (*DoS*)

Worst-Case Lower Bounds

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- Integer Transition Systems

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  - detect decreasing loops (JAR 17)

    $\Rightarrow$ much more efficient and applicable to most examples

Decreasing Loops

Generalizing loops to prove

lower bounds for $\mathsf{rc}(n)$.

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- linear and

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rc(n): Length of longest derivation starting with a basic term of size $\leq n$

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Example: il

$\mathrm{rc}(n)$: Length of longest derivation starting with a basic term of size $\leq n$

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Example: il$(5, [\,]) \to^* [0, 1, 2, 3, 4]$

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Example: $\mathsf{il}(5, [\,]) \to^* [0, 1, 2, 3, 4]$

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Here: $\mathsf{rc}(n) \in \Omega(n)$

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Here: $\mathsf{rc}(n) \in \Omega(n)$ and $\mathsf{rc}(n) \in \mathcal{O}(n)$

Loops

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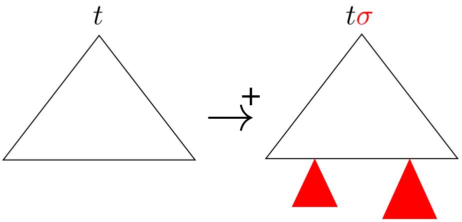

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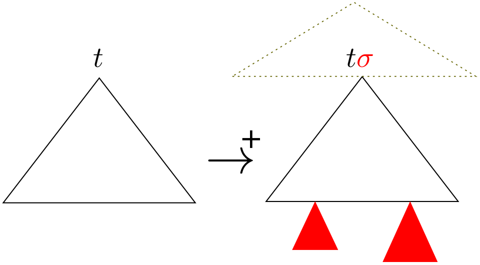

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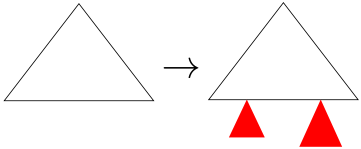

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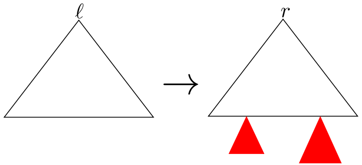

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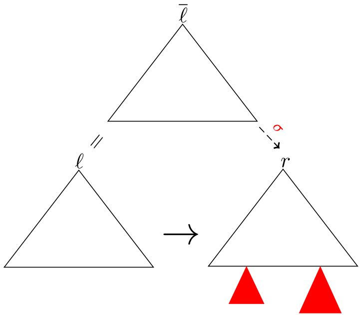

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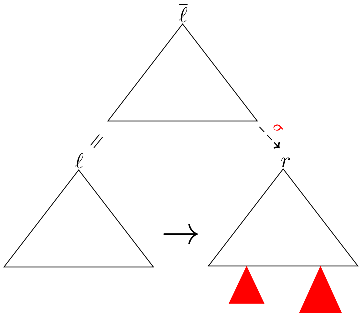

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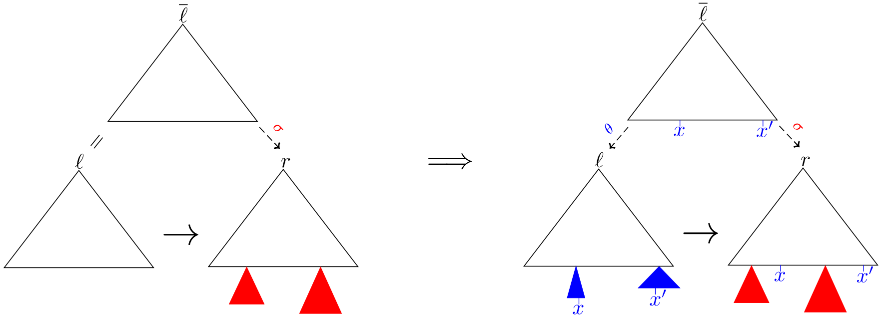

Generalizing Loops

$\mathsf{il}(\mathsf{s}(x), ys) \;\to\; \mathsf{il}(x, \mathsf{cons}(x, ys))$

$$\mathsf{il}(s(x), ys) \xleftarrow{\{x/\mathsf{s}(x)\}} \mathsf{il}(x, ys) \xrightarrow{\{ys/\mathsf{cons}(x,ys)\}} \mathsf{il}(x, \mathsf{cons}(x, ys))$$

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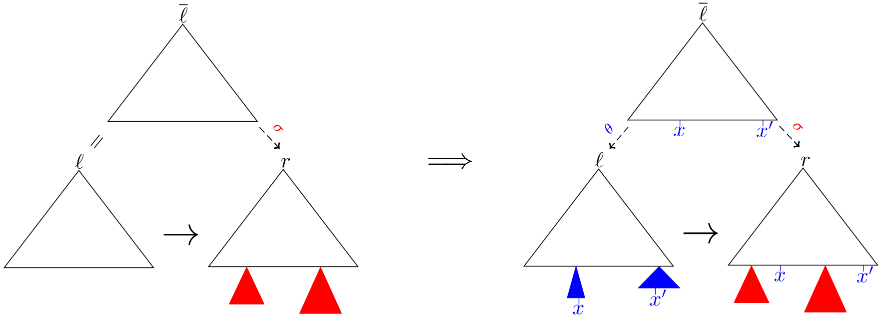

θ: Pumping Substitution

Generalizing Loops

$\mathsf{il}(\mathsf{s}(x), ys) \;\to\; \mathsf{il}(x, \mathsf{cons}(x, ys))$

$$\mathsf{il}(x, ys)$$

$$\{x/\mathsf{s}(x)\} \qquad \{ys/\mathsf{cons}(x,ys)\}$$

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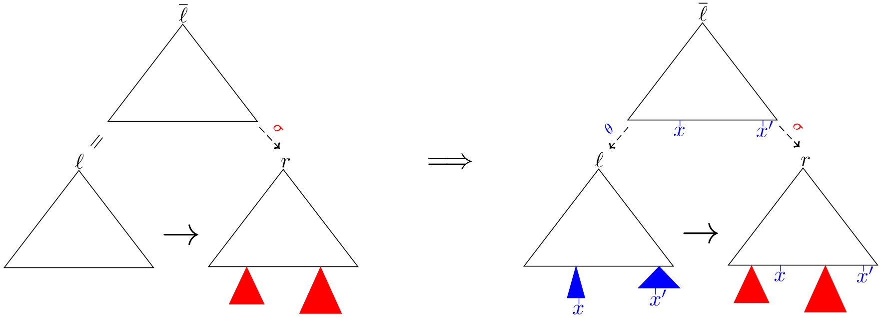

θ: Pumping Substitution $\qquad$ σ: Result Substitution

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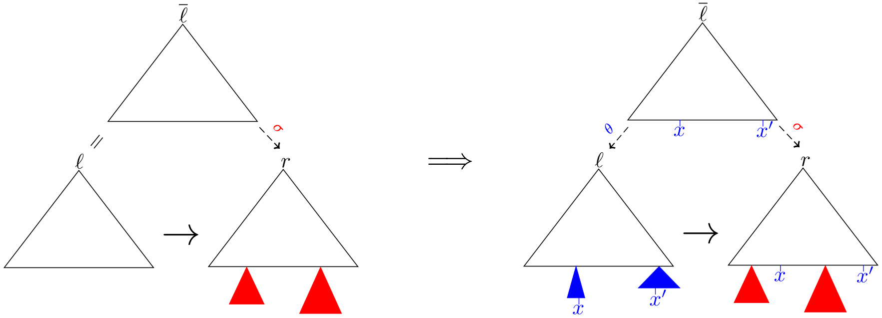

θ: Pumping Substitution σ: Result Substitution $\bar{\ell}$: Base Term

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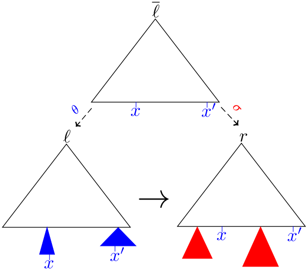
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr, "\to" description, phantom] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}$$



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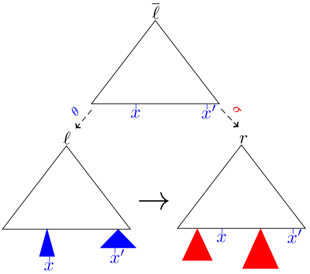

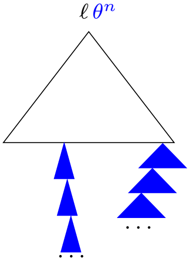

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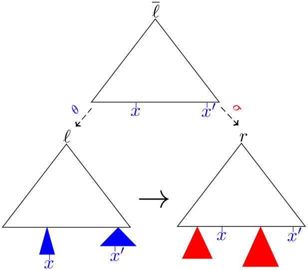
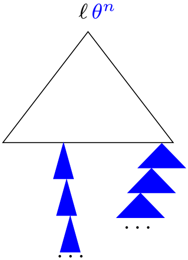
$$\ell\,\theta^n \quad = \quad \mathsf{il}(\mathsf{s}^{n+1}(x), ys)$$





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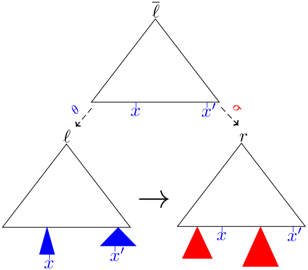
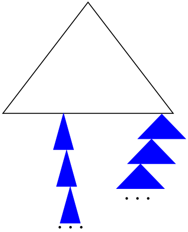


$\ell\,\theta^n \quad = \quad \mathsf{il}(\mathsf{s}^{n+1}(x), ys) \qquad \to$

$r\,\theta^n$

$\ell\,\theta^n$



$r\,\theta^n$

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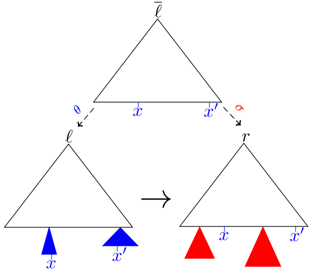
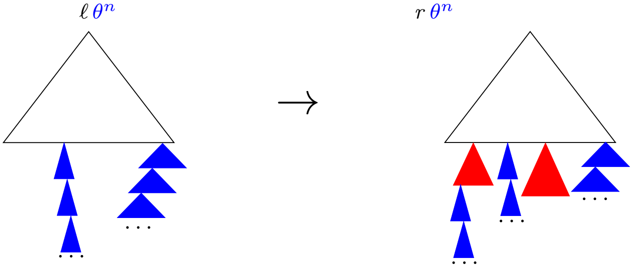
\end{aligned} \quad \rightarrow$$





Generalizing Loops

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$$
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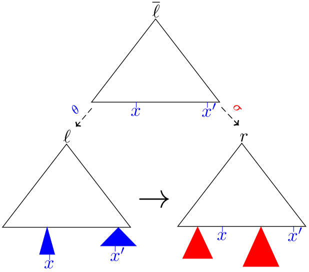
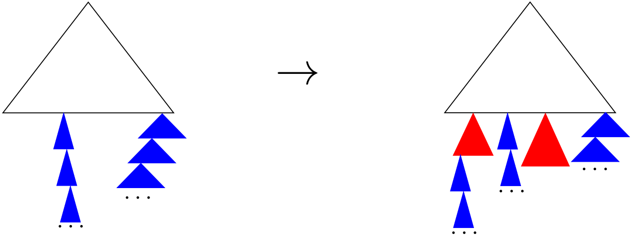
\end{aligned}
\quad \rightarrow
$$



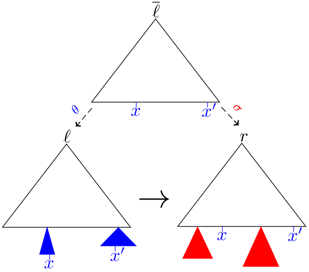
$\ell\,\theta^n$ $\qquad$ $r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n$



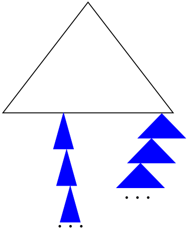
Generalizing Loops

$$\begin{tikzcd}[column sep=small] & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\ \mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys)) \end{tikzcd}$$

$$\begin{aligned} \ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) \\ r\,\theta^n &= \mathsf{il}(\mathsf{s}^{n}(x),\, \mathsf{cons}(\mathsf{s}^{n}(x),\, ys)) \end{aligned} \quad \rightarrow$$

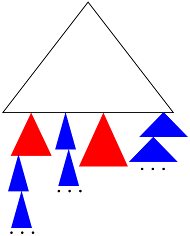


$\ell\,\theta^n$



$\longrightarrow$

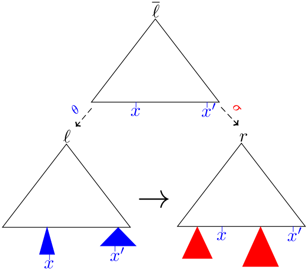
$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta$



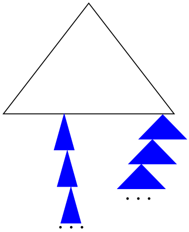
Generalizing Loops

$$
\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}
$$

$$
\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) \\
r\,\theta^n &= \mathsf{il}(\mathsf{s}^{n}(x),\, \mathsf{cons}(\mathsf{s}^{n}(x),\, ys))
\end{aligned}
\quad \rightarrow
$$

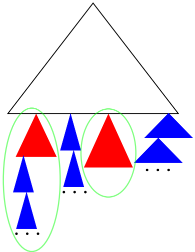


$\ell\,\theta^n$



$\rightarrow$

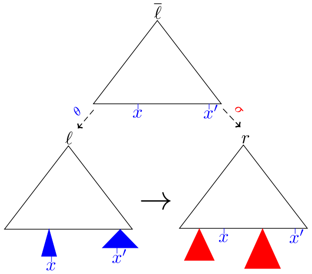
$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta$



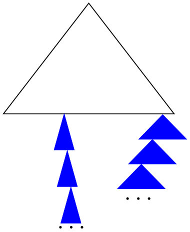
Generalizing Loops

$$
\begin{tikzcd}
 & \mathsf{il}(\textcolor{blue}{x},\textcolor{red}{ys}) \arrow[dl, dashed, "\textcolor{blue}{\{x/\mathsf{s}(x)\}}"'] \arrow[dr, dashed, "\textcolor{red}{\{ys/\mathsf{cons}(x,ys)\}}"] & \\
\mathsf{il}(\textcolor{blue}{\mathsf{s}(x)},ys) \arrow[rr] & & \mathsf{il}(\textcolor{blue}{x},\textcolor{red}{\mathsf{cons}(x,ys)})
\end{tikzcd}
$$

$$
\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) \\
r\,\theta^n &= \mathsf{il}(\mathsf{s}^{n}(x),\, \mathsf{cons}(\mathsf{s}^{n}(x),\, ys))
\end{aligned}
\qquad \to
$$

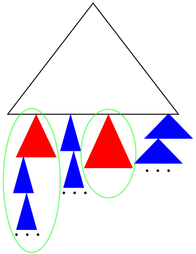


$\ell\,\theta^n$



$\to$

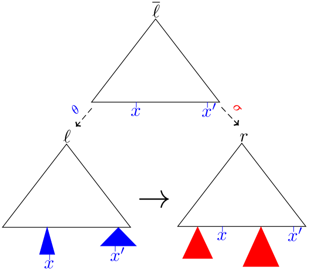
$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta = \ell\,\theta^{n-1}\,\delta$



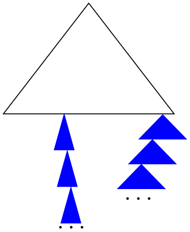
Generalizing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}$$

$$\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) \\
\ell\,\theta^{n-1}\,\delta &= \mathsf{il}(\mathsf{s}^n(x),\, \mathsf{cons}(\mathsf{s}^n(x),\, ys))
\end{aligned}
\quad \rightarrow$$

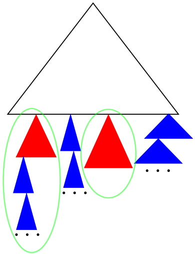


$\ell\,\theta^n$



$\longrightarrow$

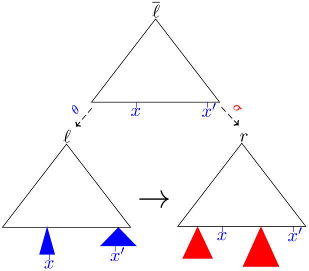
$$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta = \ell\,\theta^{n-1}\,\delta$$



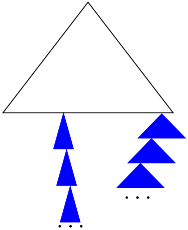
Generalizing Loops

$$
\begin{tikzcd}
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "{\{x/\mathsf{s}(x)\}}"'] \arrow[dr, dashed, "{\{ys/\mathsf{cons}(x,ys)\}}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}
$$

$$
\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) \\
\ell\,\theta^{n-1}\,\delta &= \mathsf{il}(\mathsf{s}^n(x),\, \mathsf{cons}(\mathsf{s}^n(x),\, ys))
\end{aligned}
\quad \rightarrow
$$

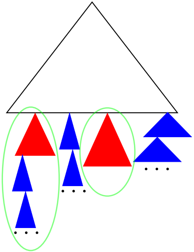


$\ell\,\theta^n$



$\longrightarrow$

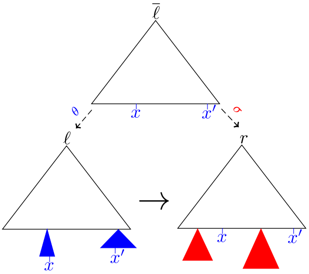
$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta = \ell\,\theta^{n-1}\,\delta$



$\longrightarrow \quad \dots$

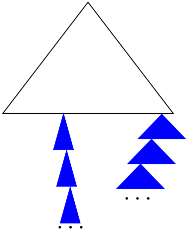
Generalizing Loops

$$
\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x, ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}
$$



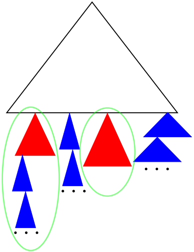
$$
\begin{array}{lcll}
\ell\,\theta^n & = & \mathsf{il}(\mathsf{s}^{n+1}(x), ys) & \to \\
\ell\,\theta^{n-1}\,\delta & = & \mathsf{il}(\mathsf{s}^{n}(x), \mathsf{cons}(\mathsf{s}^{n}(x), ys)) & \to \\
 & & \mathsf{il}(\mathsf{s}^{n-1}(x), \mathsf{cons}(\mathsf{s}^{n-1}(x), \ldots)) &
\end{array}
$$

$\ell\,\theta^n$



$\longrightarrow$

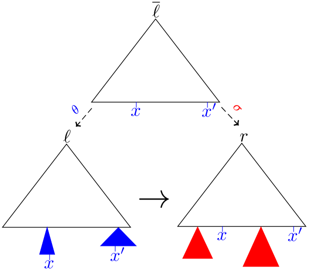
$r\,\theta^n \;=\; \bar{\ell}\,\sigma\,\theta^n \;=\; \bar{\ell}\,\theta^n\,\delta \;=\; \ell\,\theta^{n-1}\,\delta$



$\longrightarrow \qquad \ldots$

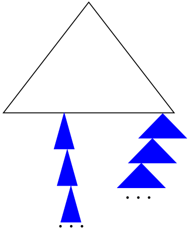
Generalizing Loops

$$
\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}
$$



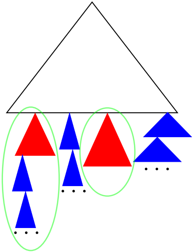
$$
\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x),\, ys) && \rightarrow \\
\ell\,\theta^{n-1}\,\delta &= \mathsf{il}(\mathsf{s}^{n}(x),\, \mathsf{cons}(\mathsf{s}^{n}(x),\, ys)) && \rightarrow \\
\ell\,\theta^{n-2}\,\delta' &= \mathsf{il}(\mathsf{s}^{n-1}(x),\, \mathsf{cons}(\mathsf{s}^{n-1}(x),\, \ldots))
\end{aligned}
$$

$\ell\,\theta^n$



$\longrightarrow$

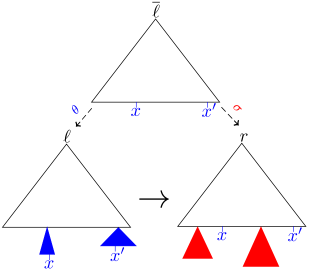
$r\,\theta^n \;=\; \bar{\ell}\,\sigma\,\theta^n \;=\; \bar{\ell}\,\theta^n\,\delta \;=\; \ell\,\theta^{n-1}\,\delta$



$\longrightarrow \quad \ldots$

Generalizing Loops

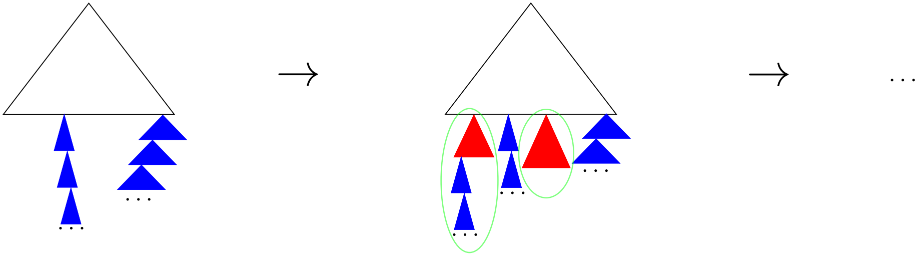
$$\begin{tikzcd}
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}$$



$$\begin{aligned}
\ell\,\theta^n &= \mathsf{il}(\mathsf{s}^{n+1}(x), ys) & \rightarrow \\
\ell\,\theta^{n-1}\,\delta &= \mathsf{il}(\mathsf{s}^n(x), \mathsf{cons}(\mathsf{s}^n(x), ys)) & \rightarrow \\
\ell\,\theta^{n-2}\,\delta' &= \mathsf{il}(\mathsf{s}^{n-1}(x), \mathsf{cons}(\mathsf{s}^{n-1}(x), \ldots)) & \rightarrow\ \ldots
\end{aligned}$$

$\ell\,\theta^n$

$r\,\theta^n = \bar{\ell}\,\sigma\,\theta^n = \bar{\ell}\,\theta^n\,\delta = \ell\,\theta^{n-1}\,\delta$

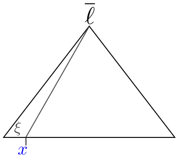


Decreasing Loops

$$
\begin{tikzcd}
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x, ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}
$$

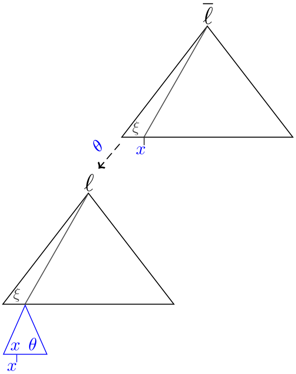
Decreasing Loops

$$\begin{array}{ccc} & \mathsf{il}(x,ys) & \\ \{x/\mathsf{s}(x)\} \swarrow & & \searrow \{ys/\mathsf{cons}(x,ys)\} \\ \mathsf{il}(\mathsf{s}(x),ys) & \to & \mathsf{il}(x,\mathsf{cons}(x,ys)) \end{array}$$



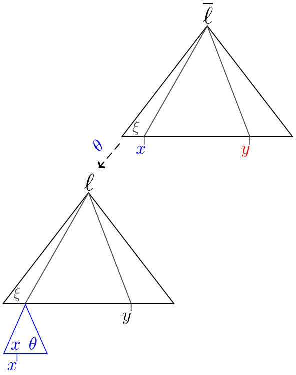
Decreasing Loops

$$\begin{tikzcd}
& \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}$$



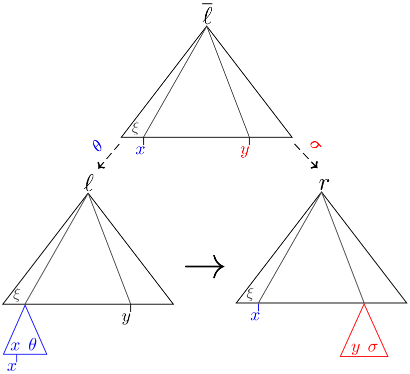
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x),ys) \arrow[rr] & & \mathsf{il}(x,\mathsf{cons}(x,ys))
\end{tikzcd}$$



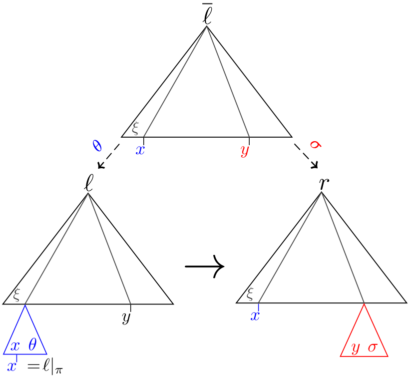
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
& \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}$$



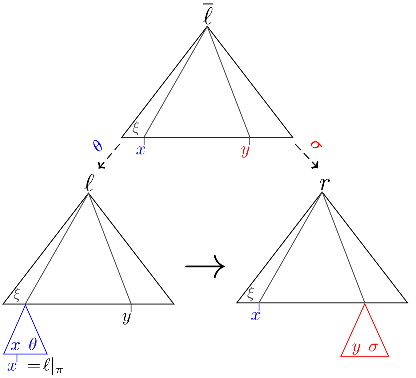
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\mathsf{s}(x), ys) \arrow[rr, "\to" description, phantom] & & \mathsf{il}(x, \mathsf{cons}(x, ys))
\end{tikzcd}$$



Decreasing Loops

$$\begin{tikzcd} & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\ \mathsf{il}(\mathsf{s}(\underset{\substack{\uparrow\\ \pi}}{x}),ys) \arrow[rr] & & \mathsf{il}(\underset{\substack{\uparrow\\ \xi}}{x},\mathsf{cons}(x,ys)) \end{tikzcd}$$



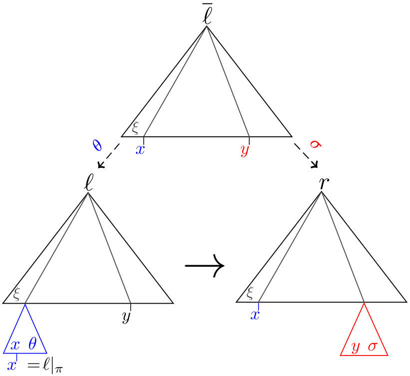
Decreasing Loops

$$\begin{tikzcd} & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\ \mathsf{il}(\mathsf{s}(\underset{\pi}{\underset{\uparrow}{x}}),ys) \arrow[rr] & & \mathsf{il}(\underset{\xi}{\underset{\uparrow}{x}},\mathsf{cons}(x,ys)) \end{tikzcd}$$

Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic



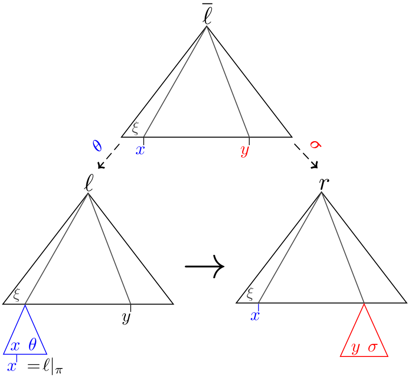
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\underset{\underset{\pi}{\uparrow}}{\mathsf{s}(x)}, ys) \arrow[rr] & & \mathsf{il}(\underset{\underset{\xi}{\uparrow}}{x}, \mathsf{cons}(x, ys))
\end{tikzcd}$$

Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi = x$



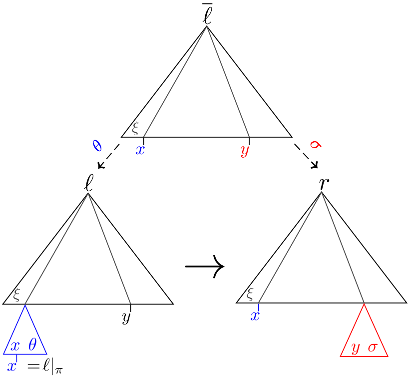
Decreasing Loops

$$\begin{array}{ccc} & \mathsf{il}(x,ys) & \\ \swarrow_{\{x/\mathsf{s}(x)\}} & & \searrow^{\{ys/\mathsf{cons}(x,ys)\}} \\ \mathsf{il}(\mathsf{s}(\underset{\pi}{\underset{\uparrow}{x}}),ys) & \to & \mathsf{il}(\underset{\xi}{\underset{\uparrow}{x}},\mathsf{cons}(x,ys)) \end{array}$$

Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi = x$
- $r|_\xi = x$ for some $\xi < \pi$



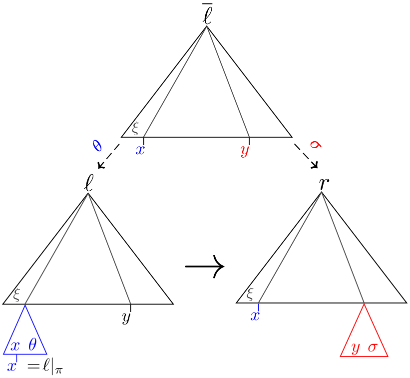
Decreasing Loops

$$\begin{array}{ccc} & \mathsf{il}(x,ys) & \\ \swarrow_{\{x/\mathsf{s}(x)\}} & & \searrow^{\{ys/\mathsf{cons}(x,ys)\}} \\ \mathsf{il}(\underset{\substack{\uparrow \\ \pi}}{\mathsf{s}(x)},ys) & \to & \mathsf{il}(\underset{\substack{\uparrow \\ \xi}}{x},\mathsf{cons}(x,ys)) \end{array}$$

Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi = x$
- $r|_\xi = x$ for some $\xi < \pi$
- $\overline{\ell} = \ell[x]_\xi$ matches r



Decreasing Loops

$$\mathsf{il}(x, ys) \quad \text{with substitutions } \{x/\mathsf{s}(x)\} \text{ and } \{ys/\mathsf{cons}(x, ys)\}$$

$$\underset{\pi}{\mathsf{il}(\mathsf{s}(\overset{\uparrow}{x}), ys)} \to \underset{\xi}{\mathsf{il}(\overset{\uparrow}{x}, \mathsf{cons}(x, ys))}$$

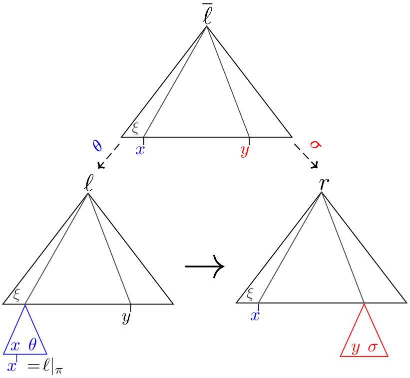
Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi = x$
- $r|_\xi = x$ for some $\xi < \pi$
- $\overline{\ell} = \ell[x]_\xi$ matches r

Theorem: Linear Bounds

If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.



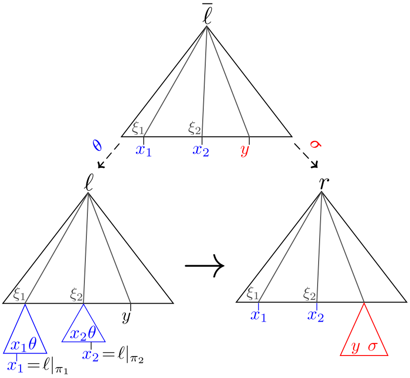
Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\ \mathsf{il}(\mathsf{s}(\underset{\substack{\uparrow\\ \pi}}{x}), ys) & \rightarrow & \mathsf{il}(\underset{\substack{\uparrow\\ \xi}}{x}, \mathsf{cons}(x, ys)) \end{tikzcd}$$

Decreasing Loop $\ell \to \quad r$

$x \qquad \in \mathcal{V}$, $\pi \qquad \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi \ = x$
- $r|_\xi \ = x$ for some $\xi \ < \pi$
- $\overline{\ell} = \ell\,[x\ ]_\xi \qquad$ matches r



Theorem: Linear Bounds

If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

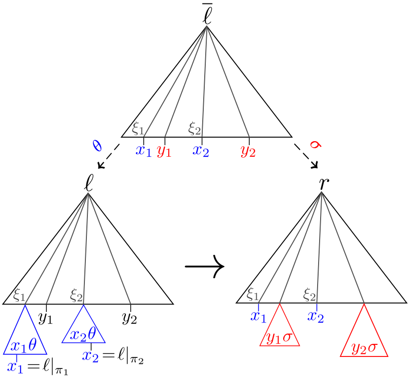
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x, ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x, ys)\}"] & \\
\mathsf{il}(\mathsf{s}(\underset{\substack{\uparrow\\ \pi}}{x}), ys) \arrow[rr, "\to", phantom] & & \mathsf{il}(\underset{\substack{\uparrow\\ \xi}}{x}, \mathsf{cons}(x, ys))
\end{tikzcd}$$

Decreasing Loop $\ell \to r$

$x \in \mathcal{V}$, $\pi \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_\pi = x$
- $r|_\xi = x$ for some $\xi < \pi$
- $\overline{\ell} = \ell[x]_\xi$ matches r



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\mathsf{il}(\mathsf{s}(\underset{\substack{\uparrow \\ \pi}}{x}),ys) \arrow[rr] & & \mathsf{il}(\underset{\substack{\uparrow \\ \xi}}{x},\mathsf{cons}(x,ys))
\end{tikzcd}$$

Decreasing Loop $\ell \to \quad r$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_{\pi} \; = x$
- $r|_{\xi} \; = x$ for some $\xi \; < \pi$
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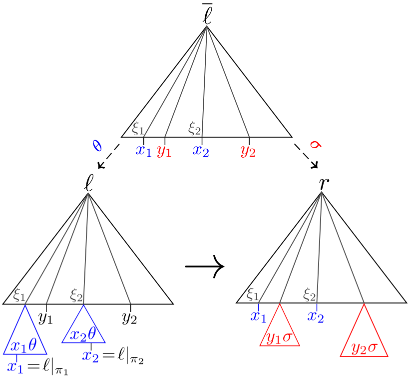

Decreasing Loops

$$\mathsf{il}(x, ys)$$

$\{x/\mathsf{s}(x)\}$ ↙ ↘ $\{ys/\mathsf{cons}(x,ys)\}$

$$\mathsf{il}(\mathsf{s}(\underset{\uparrow \atop \pi}{x}), ys) \quad \to \quad \mathsf{il}(\underset{\uparrow \atop \xi}{x}, \mathsf{cons}(x, ys))$$

Decreasing Loop $\ell \to \quad r$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

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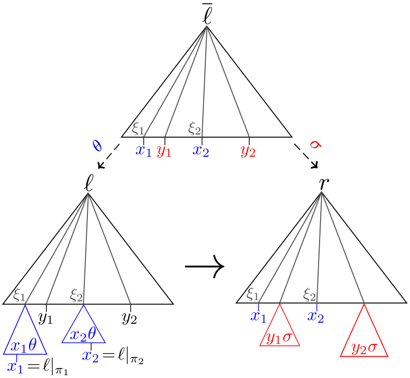

Decreasing Loops

$$\begin{array}{ccc} & \mathsf{il}(x, ys) & \\ \swarrow^{\{x/\mathsf{s}(x)\}} & & \searrow^{\{ys/\mathsf{cons}(x,ys)\}} \\ \mathsf{il}(\mathsf{s}(\underset{\substack{\uparrow \\ \pi}}{x}), ys) & \rightarrow & \mathsf{il}(\underset{\substack{\uparrow \\ \xi}}{x}, \mathsf{cons}(x, ys)) \end{array}$$

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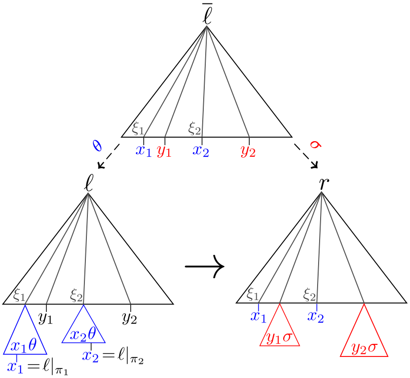

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Decreasing Loop $\ell \to r$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

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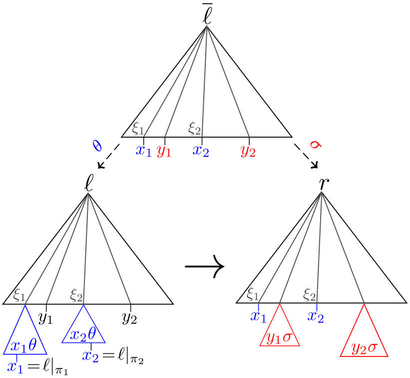
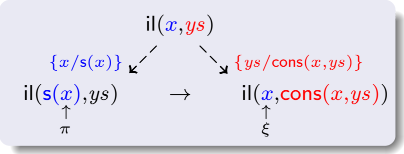


Decreasing Loops



Decreasing Loop $\ell \to r$

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- $\ell|_{\pi_i} = x_i$
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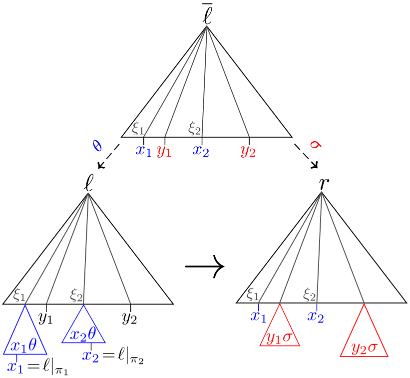


Theorem: Linear Bounds

If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

Decreasing Loops?

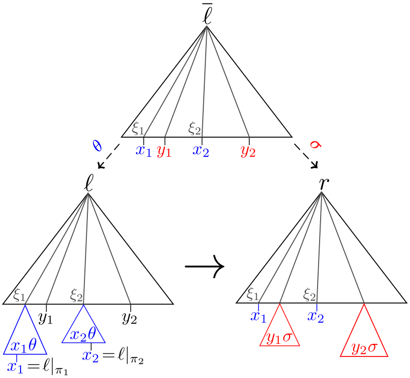
- $\mathsf{f}(\mathsf{s}(x), x) \;\to\; \mathsf{f}(x, x)$

Decreasing Loops

$\mathsf{il}(x, ys)$

$\{x/\mathsf{s}(x)\}$ $\{ys/\mathsf{cons}(x,ys)\}$

$\underset{\pi}{\mathsf{il}(\mathsf{s}(x), ys)} \rightarrow \underset{\xi}{\mathsf{il}(x, \mathsf{cons}(x, ys))}$



Decreasing Loop $\ell \rightarrow r$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

- ℓ basic
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\bar{\ell} = \ell\,[x_1]_{\xi_1} \ldots [x_m]_{\xi_m}$ matches r

Theorem: Linear Bounds

If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x), x) \rightarrow \mathsf{f}(x, x)$ ✘

Decreasing Loops

$$\mathsf{il}(x, ys)$$

$\{x/\mathsf{s}(x)\}$ $\{ys/\mathsf{cons}(x,ys)\}$

$$\mathsf{il}(\mathsf{s}(\underset{\pi}{\underset{\uparrow}{x}}), ys) \to \mathsf{il}(\underset{\xi}{\underset{\uparrow}{x}}, \mathsf{cons}(x, ys))$$

Decreasing Loop $\ell \to r$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

- ℓ basic and linear
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\overline{\ell} = \ell\,[x_1]_{\xi_1} \ldots [x_m]_{\xi_m}$ matches r

Theorem: Linear Bounds

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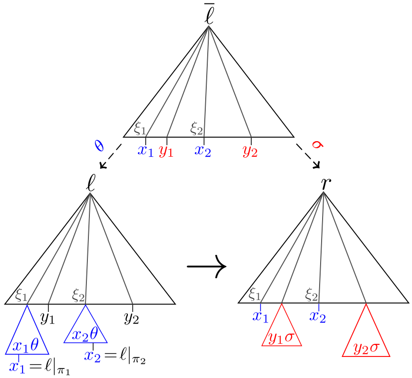

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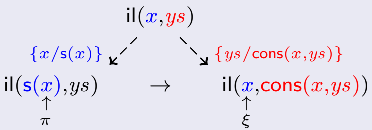
- $\mathsf{f}(\mathsf{s}(x), x) \to \mathsf{f}(x, x)$ ✘

Decreasing Loops



Decreasing Loop $\ell \to r$

$x_1, \dots, x_m \in \mathcal{V}$, $\pi_1, \dots, \pi_m \in \mathcal{P}os$ with

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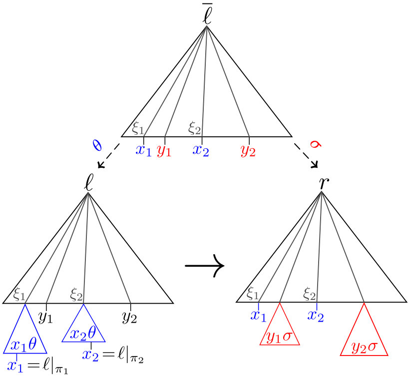

Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x), x) \to \mathsf{f}(x, x)$ ✘
- $\mathsf{f}(x) \to x$

Decreasing Loops

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$$\mathsf{il}(\underset{\substack{\uparrow\\ \pi}}{\mathsf{s}(x)}, ys) \quad \to \quad \mathsf{il}(\underset{\substack{\uparrow\\ \xi}}{x}, \mathsf{cons}(x, ys))$$

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$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

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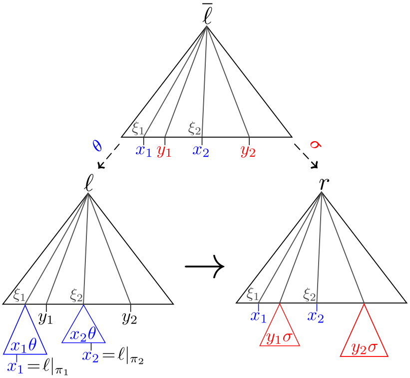

Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x), x) \to \mathsf{f}(x, x)$ ✘
- $\mathsf{f}(x) \to x$ ✘

Decreasing Loops

$\mathsf{il}(x,ys)$

$\{x/\mathsf{s}(x)\}$ $\quad$ $\{ys/\mathsf{cons}(x,ys)\}$

$\underset{\pi}{\uparrow}$ $\mathsf{il}(\mathsf{s}(x),ys) \quad \rightarrow \quad \mathsf{il}(x,\mathsf{cons}(x,ys))$ $\underset{\xi}{\uparrow}$

Decreasing Loop $\ell \rightarrow r$

$x_1,\ldots,x_m \in \mathcal{V}$, $\pi_1,\ldots,\pi_m \in \mathcal{P}os$ with

- ℓ basic and linear, $r \notin \mathcal{V}$
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\overline{\ell} = \ell\,[x_1]_{\xi_1}\ldots[x_m]_{\xi_m}$ matches r

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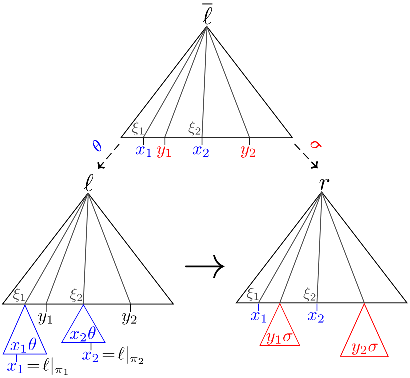


Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x),x) \rightarrow \mathsf{f}(x,x)$ ✘
- $\mathsf{f}(x) \rightarrow x$ ✘

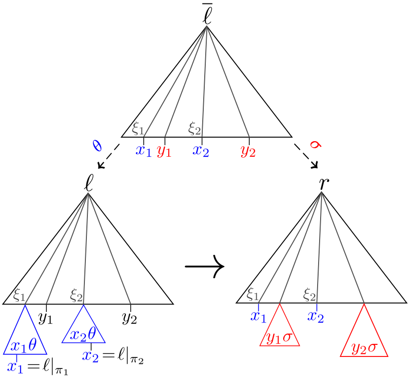
Decreasing Loops

$$\begin{tikzcd}[column sep=small]
 & \mathsf{il}(x,ys) \arrow[dl, dashed, "\{x/\mathsf{s}(x)\}"'] \arrow[dr, dashed, "\{ys/\mathsf{cons}(x,ys)\}"] & \\
\mathsf{il}(\underset{\uparrow \atop \pi}{\mathsf{s}(x)},ys) & \to & \mathsf{il}(\underset{\uparrow \atop \xi}{x},\mathsf{cons}(x,ys))
\end{tikzcd}$$

Decreasing Loop $\ell \to C[r]$

$x_1,\ldots,x_m \in \mathcal{V}$, $\pi_1,\ldots,\pi_m \in \mathcal{P}os$ with

- ℓ basic and linear, $r \notin \mathcal{V}$
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\overline{\ell} = \ell\,[x_1]_{\xi_1} \ldots [x_m]_{\xi_m}$ matches r



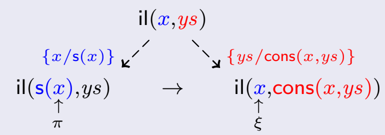
Theorem: Linear Bounds

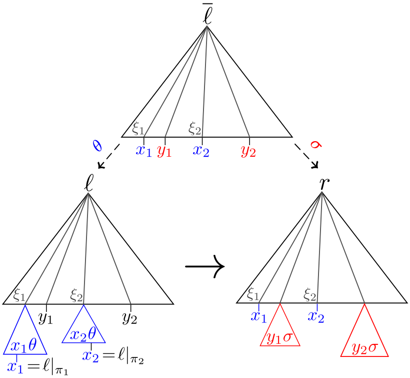
If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x),x) \to \mathsf{f}(x,x)$ ✘
- $\mathsf{f}(x) \to x$ ✘

Decreasing Loops





Decreasing Loop $\ell \to^+ C[r]$

$x_1, \ldots, x_m \in \mathcal{V}$, $\pi_1, \ldots, \pi_m \in \mathcal{P}os$ with

- ℓ basic and linear, $r \notin \mathcal{V}$
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\overline{\ell} = \ell\,[x_1]_{\xi_1} \ldots [x_m]_{\xi_m}$ matches r

Theorem: Linear Bounds

If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

Decreasing Loops?

- $\mathsf{f}(\mathsf{s}(x), x) \ \to\ \mathsf{f}(x, x)$ ✘
- $\mathsf{f}(x) \ \to\ x$ ✘

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

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context around variable x is removed in each step,
same rewrite rule again applicable to rhs

$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{f}(\mathsf{s}(x))$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

context around variable x is removed in each step,
same rewrite rule again applicable to rhs

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)) \qquad\qquad)$$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

context around variable x is removed in each step, same rewrite rule again applicable to rhs

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\rightarrow\; \mathsf{plus}(\qquad\qquad \mathsf{f}(x)\,)$$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\qquad\qquad \mathsf{f}(x)\,)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)),\; \mathsf{f}(x)\,)$$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

Fibonacci

$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)),\; \mathsf{f}(x)\,)$

Exponential Bounds

decreasing loop:	*linear bound* $\Omega(n)$
d parallel decreasing loops:	*exponential bound* $\Omega(d^n)$

Fibonacci

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)),\ \mathsf{f}(x)\,)$$

$$\begin{array}{ccccccc}
 & & & & \mathsf{f}(\mathsf{s}^{n-2}(0)) & \ldots \\
 & & & \nearrow & & \\
 & & \mathsf{f}(\mathsf{s}^{n-1}(0)) & \to & \mathsf{f}(\mathsf{s}^{n-3}(0)) & \ldots \\
 & \nearrow & & & & \\
\mathsf{f}(\mathsf{s}^{n}(0)) & & & & & \\
 & \searrow & & & & \\
 & & \mathsf{f}(\mathsf{s}^{n-2}(0)) & \to & \mathsf{f}(\mathsf{s}^{n-3}(0)) & \ldots \\
 & & & \searrow & & \\
 & & & & \mathsf{f}(\mathsf{s}^{n-4}(0)) & \ldots
\end{array}$$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

Fibonacci

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \;\to\; \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)),\; \mathsf{f}(x)\,)$$

$$
\begin{array}{ccccc}
 & & & & \mathsf{f}(\mathsf{s}^{n-2}(0)) \quad \dots \\
 & & & \nearrow & \\
 & & \mathsf{f}(\mathsf{s}^{n-1}(0)) & \rightarrow & \mathsf{f}(\mathsf{s}^{n-3}(0)) \quad \dots \\
 & \nearrow & & & \\
\mathsf{f}(\mathsf{s}^{n}(0)) & & & & \\
 & \searrow & & & \\
 & & \mathsf{f}(\mathsf{s}^{n-2}(0)) & \rightarrow & \mathsf{f}(\mathsf{s}^{n-3}(0)) \quad \dots \\
 & & & \searrow & \\
 & & & & \mathsf{f}(\mathsf{s}^{n-4}(0)) \quad \dots
\end{array}
$$

Here: $\mathsf{rc}(n) \in \Omega(2^n)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

Traverse

$\mathsf{tr}(\mathsf{node}(x, y)) \to \mathsf{node}(\mathsf{tr}(x), \mathsf{tr}(y))$

$$
\begin{array}{ccccccc}
 & & & & \mathsf{f}(\mathsf{s}^{n-2}(0)) & & \ldots \\
 & & & \nearrow & & & \\
 & & \mathsf{f}(\mathsf{s}^{n-1}(0)) & \to & \mathsf{f}(\mathsf{s}^{n-3}(0)) & & \ldots \\
 & \nearrow & & & & & \\
\mathsf{f}(\mathsf{s}^{n}(0)) & & & & & & \\
 & \searrow & & & & & \\
 & & \mathsf{f}(\mathsf{s}^{n-2}(0)) & \to & \mathsf{f}(\mathsf{s}^{n-3}(0)) & & \ldots \\
 & & & \searrow & & & \\
 & & & & \mathsf{f}(\mathsf{s}^{n-4}(0)) & & \ldots
\end{array}
$$

Here: $\mathsf{rc}(n) \in \Omega(2^n)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

Traverse

$$\mathsf{tr}(\mathsf{node}(x, y)) \to \mathsf{node}(\mathsf{tr}(x), \mathsf{tr}(y))$$

$$\begin{array}{ccccc}
 & & & \nearrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & \nearrow & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
\mathsf{tr}(\mathsf{node}^{m}(x,x)) & & & & \\
 & \searrow & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & \searrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots
\end{array}$$

Here: $\mathsf{rc}(n) \in \Omega(2^n)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

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Traverse

$\mathsf{tr}(\mathsf{node}(x, y)) \to \mathsf{node}(\mathsf{tr}(x), \mathsf{tr}(y))$

$$\begin{array}{ccccccc}
 & & & & & \nearrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & \nearrow & & & & \\
\mathsf{tr}(\mathsf{node}^{m}(x,x)) & & & & & & \\
 & & \searrow & & & & \\
 & & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & & & \searrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots
\end{array}$$

Here: $\mathsf{rc}(n) \notin \Omega(2^n)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$

Traverse

$\mathsf{tr}(\mathsf{node}(x, y)) \to \mathsf{node}(\mathsf{tr}(x), \mathsf{tr}(y))$

$$
\begin{array}{ccccccc}
 & & & & & & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & & & \nearrow & \\
 & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & & & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & \nearrow & & & & & \\
\mathsf{tr}(\mathsf{node}^{m}(x,x)) & & & & & & \\
 & \searrow & & & & & \\
 & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & & & \to & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & & & \searrow & \\
 & & & & & & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots
\end{array}
$$

Here: $\mathsf{rc}(n) \notin \Omega(2^n),\ \mathsf{rc}(n) \in \mathcal{O}(n)$

Exponential Bounds

decreasing loop: *linear bound* $\Omega(n)$

d parallel decreasing loops: *exponential bound* $\Omega(d^n)$ **?**

Traverse

$\mathsf{tr}(\mathsf{node}(x, y)) \to \mathsf{node}(\mathsf{tr}(x), \mathsf{tr}(y))$

$$
\begin{array}{ccccc}
 & & & \nearrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & \rightarrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & \nearrow & & & \\
\mathsf{tr}(\mathsf{node}^{m}(x,x)) & & & & \\
 & \searrow & & & \\
 & & \mathsf{tr}(\mathsf{node}^{m-1}(x,x)) & \rightarrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots \\
 & & & \searrow & \mathsf{tr}(\mathsf{node}^{m-2}(x,x)) \quad \ldots
\end{array}
$$

Here: $\mathsf{rc}(n) \notin \Omega(2^n)$, $\mathsf{rc}(n) \in \mathcal{O}(n)$

Compatible Decreasing Loops

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

Compatible Decreasing Loops

$$\begin{tikzcd} & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr] & & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

Compatible Decreasing Loops

$$\begin{array}{ccc} & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) & \\ \swarrow^{\theta} & & \searrow^{\varnothing} \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C[\mathsf{f}(\mathsf{s}(x))] \end{array}$$

$$\begin{array}{ccc} & \overline{\ell}' = \mathsf{f}(x) & \\ \swarrow^{\theta'} & & \searrow^{\varnothing} \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C'[\mathsf{f}(x)] \end{array}$$

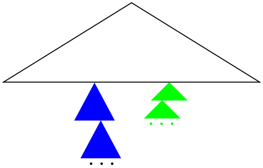
$$\mathsf{f}(\ \mathsf{s}(\mathsf{s}(x))\ ) \to \mathsf{plus}(\ \mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\ )$$

Compatible Decreasing Loops

$$
\begin{tikzcd}
 & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\
\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, "" description, phantom, "\to"] & & C[\mathsf{f}(\mathsf{s}(x))]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \overline{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\
\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, phantom, "\to"] & & C'[\mathsf{f}(x)]
\end{tikzcd}
$$

$\ell\,\theta^n\,\theta'^n$



$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

Compatible Decreasing Loops

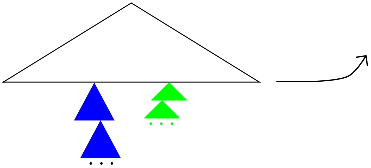
$$\begin{tikzcd}[column sep=small] & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C'[\mathsf{f}(x)] \end{tikzcd}$$

$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$

$r\,\theta^n\,\theta'^n$

$\ell\,\theta^n\,\theta'^n$



Compatible Decreasing Loops

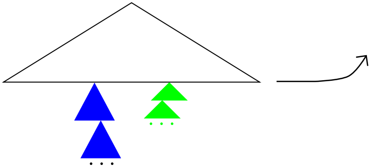
$$\begin{tikzcd} & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr] & & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr] & & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

$$r\,\theta^n\,\theta'^n \;=\; \overline{\ell}\,\sigma\,\theta^n\,\theta'^n$$

$\ell\,\theta^n\,\theta'^n$

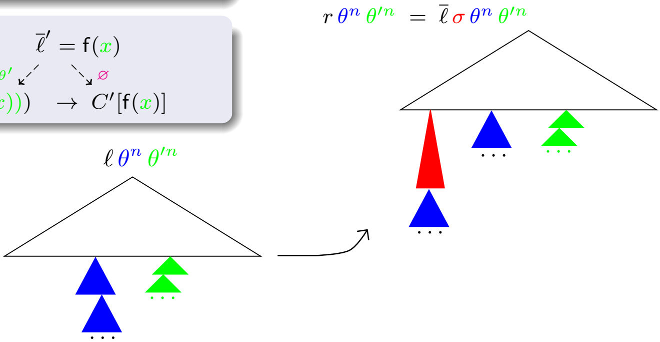


Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr] & & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr] & & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

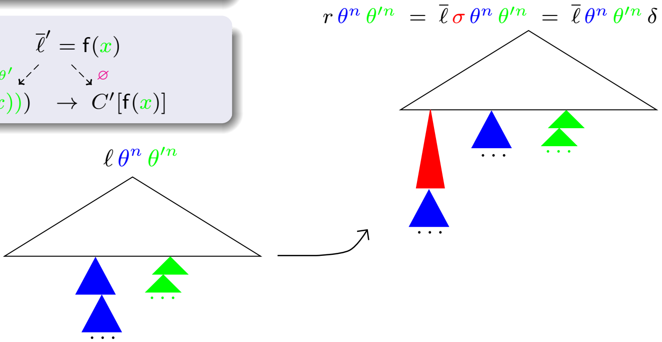


Compatible Decreasing Loops

$$\begin{array}{ccc} & \bar\ell = \mathsf{f}(\mathsf{s}(x)) & \\ \swarrow \theta & & \searrow \varnothing \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C[\mathsf{f}(\mathsf{s}(x))] \end{array}$$

$$\begin{array}{ccc} & \bar\ell' = \mathsf{f}(x) & \\ \swarrow \theta' & & \searrow \varnothing \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C'[\mathsf{f}(x)] \end{array}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$



- σ does not interfere with θ'

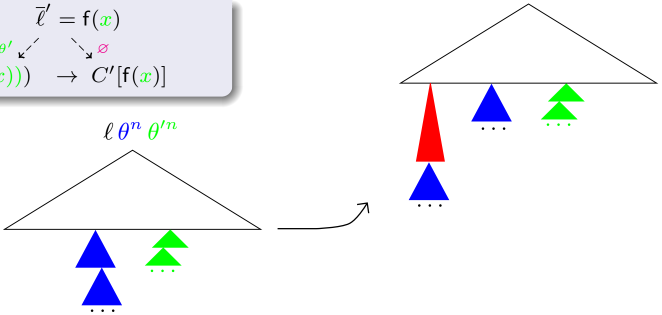
Compatible Decreasing Loops

$$\begin{array}{ccc} & \bar{\ell} = \mathsf{f}(\mathsf{s}(x)) & \\ \theta \swarrow & & \searrow \varnothing \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C[\mathsf{f}(\mathsf{s}(x))] \end{array}$$

$$\begin{array}{ccc} & \bar{\ell}' = \mathsf{f}(x) & \\ \theta' \swarrow & & \searrow \varnothing \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C'[\mathsf{f}(x)] \end{array}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

$$r\,\theta^n\,\theta'^n = \bar{\ell}\,\sigma\,\theta^n\,\theta'^n = \bar{\ell}\,\theta^n\,\theta'^n\,\delta = \ell\,\theta^{n-1}\,\theta'^n\,\delta$$



- σ does not interfere with θ'

Compatible Decreasing Loops

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

$$\bar{\ell} = \mathsf{f}(\mathsf{s}(x))$$

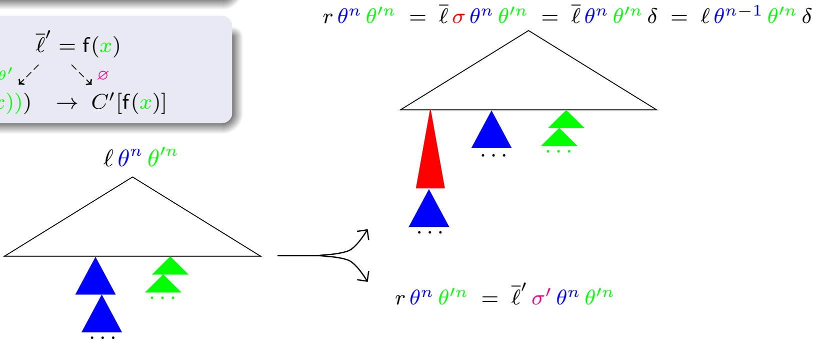
$\theta \swarrow \qquad \searrow \varnothing$

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \quad \to \quad C[\mathsf{f}(\mathsf{s}(x))]$$

$$\bar{\ell}' = \mathsf{f}(x)$$

$\theta' \swarrow \qquad \searrow \varnothing$

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \quad \to \quad C'[\mathsf{f}(x)]$$



- σ does not interfere with θ'

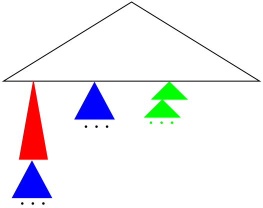
Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \overline{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \rightarrow & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

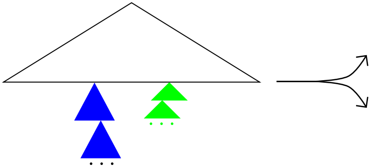
$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \rightarrow & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \rightarrow \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

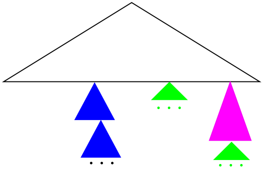
$$r\,\theta^n\,\theta'^n \;=\; \overline{\ell}\,\sigma\,\theta^n\,\theta'^n \;=\; \overline{\ell}\,\theta^n\,\theta'^n\,\delta \;=\; \ell\,\theta^{n-1}\,\theta'^n\,\delta$$



$\ell\,\theta^n\,\theta'^n$



$$r\,\theta^n\,\theta'^n \;=\; \overline{\ell}'\,\sigma'\,\theta^n\,\theta'^n$$



- σ does not interfere with θ'

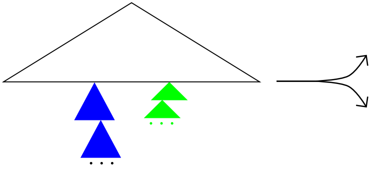
Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \bar{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

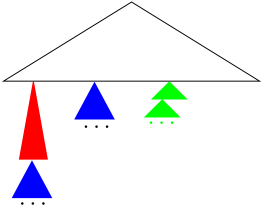
$$\begin{tikzcd}[column sep=small] & \bar{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & \to & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

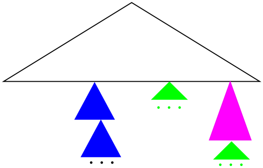
$\ell\,\theta^n\,\theta'^n$



$r\,\theta^n\,\theta'^n = \bar{\ell}\,\sigma\,\theta^n\,\theta'^n = \bar{\ell}\,\theta^n\,\theta'^n\,\delta = \ell\,\theta^{n-1}\,\theta'^n\,\delta$



$r\,\theta^n\,\theta'^n = \bar{\ell}'\,\sigma'\,\theta^n\,\theta'^n = \bar{\ell}'\,\theta^n\,\theta'^n\,\delta'$



- σ does not interfere with θ'
- σ' does not interfere with θ

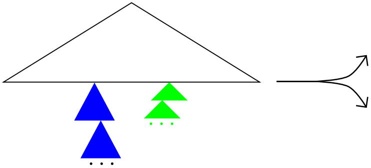
Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \bar{\ell} = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, "\to" phantom] & & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

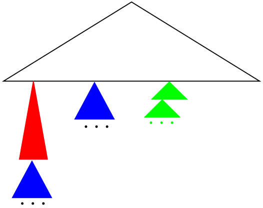
$$\begin{tikzcd}[column sep=small] & \bar{\ell}' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, "\to" phantom] & & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

$\ell\,\theta^n\,\theta'^n$

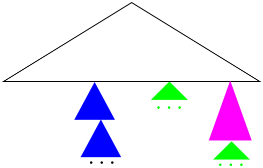


$r\,\theta^n\,\theta'^n \;=\; \bar{\ell}\,\sigma\,\theta^n\,\theta'^n \;=\; \bar{\ell}\,\theta^n\,\theta'^n\,\delta \;=\; \ell\,\theta^{n-1}\,\theta'^n\,\delta$



$r\,\theta^n\,\theta'^n \;=\; \bar{\ell}'\,\sigma'\,\theta^n\,\theta'^n \;=\; \bar{\ell}'\,\theta^n\,\theta'^n\,\delta'$

$\;=\; \bar{\ell}'\,\theta'\,\theta^n\,\theta'^{n-1}\,\delta'$



- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

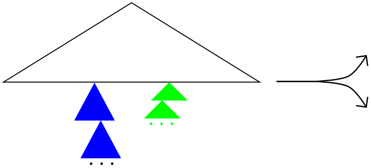
Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \bar\ell = \mathsf{f}(\mathsf{s}(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, "\to"', phantom] & & C[\mathsf{f}(\mathsf{s}(x))] \end{tikzcd}$$

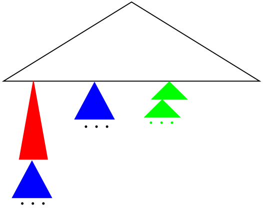
$$\begin{tikzcd}[column sep=small] & \bar\ell' = \mathsf{f}(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \arrow[rr, "\to"', phantom] & & C'[\mathsf{f}(x)] \end{tikzcd}$$

$$\mathsf{f}(\,\mathsf{s}(\mathsf{s}(x))\,) \to \mathsf{plus}(\,\mathsf{f}(\mathsf{s}(x)), \mathsf{f}(x)\,)$$

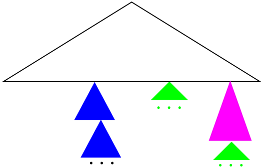
$$\ell\,\theta^n\,\theta'^n$$



$$r\,\theta^n\,\theta'^n \;=\; \bar\ell\,\sigma\,\theta^n\,\theta'^n \;=\; \bar\ell\,\theta^n\,\theta'^n\,\delta \;=\; \ell\,\theta^{n-1}\,\theta'^n\,\delta$$



$$\begin{aligned} r\,\theta^n\,\theta'^n &= \bar\ell'\,\sigma'\,\theta^n\,\theta'^n &&= \bar\ell'\,\theta^n\,\theta'^n\,\delta' \\ &= \bar\ell'\,\theta'\,\theta^n\,\theta'^{n-1}\,\delta' &&= \ell\,\theta^n\,\theta'^{n-1}\,\delta' \end{aligned}$$



Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\mathsf{tr}(\,\mathsf{node}(x, y)\,) \;\to\; \mathsf{node}(\,\mathsf{tr}(x), \,\mathsf{tr}(y)\,)$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$
\begin{tikzcd}[column sep=small]
 & \overline{\ell} = \mathsf{tr}(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\
\mathsf{tr}(\mathsf{node}(x, y)) & \to & C[\mathsf{tr}(x)]
\end{tikzcd}
$$

$\mathsf{tr}(\,\mathsf{node}(x, y)\,) \ \to\ \mathsf{node}(\,\mathsf{tr}(x), \mathsf{tr}(y)\,)$

Two decreasing loops are *compatible* iff

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- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{tikzcd} & \overline{\ell} = \mathsf{tr}(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x, y)) \arrow[rr] & & C[\mathsf{tr}(x)] \end{tikzcd}$$

$$\mathsf{tr}(\ \mathsf{node}(x, y)\ ) \ \rightarrow\ \mathsf{node}(\ \mathsf{tr}(x), \mathsf{tr}(y)\ )$$

$$\begin{tikzcd} & \overline{\ell}' = \mathsf{tr}(y) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x, y)) \arrow[rr] & & C'[\mathsf{tr}(y)] \end{tikzcd}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{array}{ccc} & \overline{\ell} = \mathsf{tr}(x) & \\ \theta \swarrow & & \searrow \varnothing \\ \mathsf{tr}(\mathsf{node}(x,y)) & \to & C[\mathsf{tr}(x)] \end{array}$$

$$\begin{array}{ccc} & \overline{\ell}' = \mathsf{tr}(y) & \\ \theta' \swarrow & & \searrow \varnothing \\ \mathsf{tr}(\mathsf{node}(x,y)) & \to & C'[\mathsf{tr}(y)] \end{array}$$

$$\mathsf{tr}(\,\mathsf{node}(x,y)\,) \;\to\; \mathsf{node}(\,\mathsf{tr}(x),\,\mathsf{tr}(y)\,)$$

$$\theta \quad = \quad \{\, x \,/\, \mathsf{node}(x, \qquad y\ ) \qquad \}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{array}{ccc} & \overline{\ell} = \mathsf{tr}(x) & \\ \theta \swarrow & & \searrow \varnothing \\ \mathsf{tr}(\mathsf{node}(x,y)) & \to & C[\mathsf{tr}(x)] \end{array}$$

$$\begin{array}{ccc} & \overline{\ell}' = \mathsf{tr}(y) & \\ \theta' \swarrow & & \searrow \varnothing \\ \mathsf{tr}(\mathsf{node}(x,y)) & \to & C'[\mathsf{tr}(y)] \end{array}$$

$$\mathsf{tr}(\,\mathsf{node}(x,y)\,) \;\to\; \mathsf{node}(\,\mathsf{tr}(x),\, \mathsf{tr}(y)\,)$$

$$\theta\,\theta' \quad = \quad \{\, x \,/\, \mathsf{node}(x, \mathsf{node}(x,y)),\; y \,/\, \mathsf{node}(x,y) \,\}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \overline{\ell} = \mathsf{tr}(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x, y)) \arrow[rr] & & C[\mathsf{tr}(x)] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = \mathsf{tr}(y) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x, y)) \arrow[rr] & & C'[\mathsf{tr}(y)] \end{tikzcd}$$

$$\mathsf{tr}(\,\mathsf{node}(x, y)\,) \;\to\; \mathsf{node}(\,\mathsf{tr}(x), \mathsf{tr}(y)\,)$$

$$\theta\,\theta' \quad = \quad \{\, x \,/\, \mathsf{node}(x, \mathsf{node}(x, y)),\; y \,/\, \mathsf{node}(x, y) \,\}$$

$$\theta' \quad = \quad \{\, y \,/\, \mathsf{node}(\; x \;, y) \,\}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{tikzcd} & \overline{\ell} = \mathsf{tr}(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x,y)) & \rightarrow & C[\mathsf{tr}(x)] \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\ell}' = \mathsf{tr}(y) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x,y)) & \rightarrow & C'[\mathsf{tr}(y)] \end{tikzcd}$$

$$\mathsf{tr}(\,\mathsf{node}(x,y)\,) \;\rightarrow\; \mathsf{node}(\,\mathsf{tr}(x),\, \mathsf{tr}(y)\,)$$

$$\theta\,\theta' \quad = \quad \{\, x \,/\, \mathsf{node}(x, \mathsf{node}(x,y)),\; y \,/\, \mathsf{node}(x,y) \,\}$$

$$\theta'\,\theta \quad = \quad \{\, x \,/\, \mathsf{node}(x,y),\; y \,/\, \mathsf{node}(\mathsf{node}(x,y), y) \,\}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

$$\begin{tikzcd}[column sep=small] & \overline{\ell} = \mathsf{tr}(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x,y)) & \rightarrow & C[\mathsf{tr}(x)] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = \mathsf{tr}(y) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\varnothing"] & \\ \mathsf{tr}(\mathsf{node}(x,y)) & \rightarrow & C'[\mathsf{tr}(y)] \end{tikzcd}$$

$$\mathsf{tr}(\,\mathsf{node}(x,y)\,) \;\rightarrow\; \mathsf{node}(\,\mathsf{tr}(x),\,\mathsf{tr}(y)\,)$$

$$\theta\,\theta' \quad = \quad \{\, x \,/\, \mathsf{node}(x, \mathsf{node}(x,y)),\; y \,/\, \mathsf{node}(x,y) \,\}$$

$$\theta'\,\theta \quad = \quad \{\, x \,/\, \mathsf{node}(x,y),\; y \,/\, \mathsf{node}(\mathsf{node}(x,y), y) \,\}$$

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Here: Decreasing loops are not compatible

Compatible Decreasing Loops

Theorem: Linear Bounds

- If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

Compatible Decreasing Loops

Theorem: Linear, Exponential Bounds

- If a TRS has a decreasing loop, then $\mathsf{rc}(n) \in \Omega(n)$.
- If a TRS has d compatible decreasing loops, then $\mathsf{rc}(n) \in \Omega(d^n)$.

Two decreasing loops are *compatible* iff

- σ does not interfere with θ'
- σ' does not interfere with θ
- θ and θ' commute

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- If a TRS has d compatible decreasing loops, then $\mathsf{rc}(n) \in \Omega(d^n)$.
- If a TRS has a decreasing loop with $\overline{\ell} = \ell$, then $\mathsf{rc}(n) \in \Omega(\omega)$.

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Experiments (865 Examples from *TPDB*)

AProVE with Rewrite Lemmas (RTA 15)

- non-trivial lower bounds for 78 %
- average runtime 24.5 s

rc(n)	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	EXP	$\Omega(\omega)$
Σ	192	572	73	14	13	1	–

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AProVE with both

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- compatible decreasing loops imply exponential lower bound
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- but still incomplete ($\mathsf{rc}(n) \in \Omega(n)$ not semi-decidable)

Completeness and Decidability

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Theorem: Incompleteness of Decreasing Loops

Consider class of linear TRSs, where all terms in rules are basic.

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Immortality undecidable

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Immortality undecidable $\implies$ $\mathsf{rc}_{\mathcal{R}_{\mathcal{M}}}(n) \in \Omega(n)$ not semi-decidable

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$$\begin{array}{rcl} \text{Immortality undecidable} & \Longrightarrow & \mathsf{rc}_{\mathcal{R}_{\mathcal{M}}}(n) \in \Omega(n) \text{ not semi-decidable} \\ & \Longrightarrow & \text{decreasing loops incomplete for linear bounds} \quad \square \end{array}$$

Experiments (865 Examples from *TPDB*)

Without Decreasing Loops

$\mathsf{rc}_{\mathcal{R}}(n)$	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	EXP	$\Omega(\omega)$
$\mathcal{O}(1)$	(34)	–	–	–	–	–	–
$\mathcal{O}(n)$	41	114	–	–	–	–	–
$\mathcal{O}(n^2)$	5	10	3	–	–	–	–
$\mathcal{O}(n^3)$	1	1	1	1	–	–	–
$\mathcal{O}(n^{>3})$	–	2	–	–	–	–	–
EXP	–	–	–	–	–	–	–
$\mathcal{O}(\omega)$	145	445	69	13	1	13	–

With Decreasing Loops

$\mathsf{rc}_{\mathcal{R}}(n)$	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	EXP	$\Omega(\omega)$
$\mathcal{O}(1)$	(34)	–	–	–	–	–	–
$\mathcal{O}(n)$	15	140	–	–	–	–	–
$\mathcal{O}(n^2)$	–	15	3	–	–	–	–
$\mathcal{O}(n^3)$	–	2	1	1	–	–	–
$\mathcal{O}(n^{>3})$	–	2	–	–	–	–	–
EXP	–	–	–	–	–	–	–
$\mathcal{O}(\omega)$	14	374	52	10	1	145	90