A Dependency Pair Framework for Innermost Complexity Analysis of Term Rewrite Systems

Jürgen Giesl

LuFG Informatik 2, RWTH Aachen University, Germany

joint work with Lars Noschinski and Fabian Emmes

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- useful for termination of programs (Java, Haskell, Prolog, . . .)

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    - modular approach based on relative rewriting
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    - modular approach based on relative rewriting
    - for *Derivational Complexity*
      (cannot exploit strength of DPs for innermost rewriting)
  - new approach: *direct* adaption of DP framework (CADE '11)
    - modular combination of different techniques
    - automated and more powerful than previous approaches

Innermost Runtime Complexity

$$\mathcal{R}: \quad \begin{array}{rcl} \mathsf{double}(0) & \to & 0 \\ \mathsf{double}(\mathsf{s}(x)) & \to & \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{array}$$

- **Derivation Height** $\mathsf{dh}(t)$**:** length of longest $\xrightarrow{\mathrm{i}}_{\mathcal{R}}$-sequence with t

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$$\iota_{\mathcal{R}} = \mathcal{P}ol_2 \text{ iff length} \in \mathcal{O}(n^2) \qquad \ldots$$

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- Example: $\iota_{\mathcal{R}} = \mathcal{P}ol_1$

Dependency Tuples

$$\begin{aligned} \mathsf{m}(x, y) &\to \mathsf{if}(\mathsf{gt}(x, y), x, y) \\ \mathsf{if}(\mathsf{true}, x, y) &\to \mathsf{s}(\mathsf{m}(\mathsf{p}(x), y)) \\ \mathsf{if}(\mathsf{false}, x, y) &\to 0 \end{aligned} \qquad \begin{aligned} \mathsf{gt}(0, k) &\to \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) &\to \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\to \mathsf{gt}(n, k) \end{aligned} \qquad \begin{aligned} \mathsf{p}(0) &\to 0 \\ \mathsf{p}(\mathsf{s}(n)) &\to n \end{aligned}$$

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  compare lhs with *all* defined subterms of rhs *at once*

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$$\begin{aligned} \mathsf{m}^\sharp(x,y) &\to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x,y),x,y),\mathsf{gt}^\sharp(x,y)) \\ \mathsf{if}^\sharp(\mathsf{true},x,y) &\to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y),\mathsf{p}^\sharp(x)) \\ \mathsf{if}^\sharp(\mathsf{false},x,y) &\to \textsc{Com}_0 \end{aligned} \qquad \begin{aligned} \mathsf{p}^\sharp(0) &\to \textsc{Com}_0 \\ \mathsf{p}^\sharp(\mathsf{s}(n)) &\to \textsc{Com}_0 \\ \mathsf{gt}^\sharp(0,k) &\to \textsc{Com}_0 \\ \mathsf{gt}^\sharp(\mathsf{s}(n),0) &\to \textsc{Com}_0 \\ \mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) &\to \textsc{Com}_1(\mathsf{gt}^\sharp(n,k)) \end{aligned}$$

Chain Trees

$\mathsf{DT}(\mathcal{R})$:

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$(\mathcal{D},\mathcal{R})$-**Chain Tree:**

Chain Trees

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$(\mathcal{D},\mathcal{R})$-Chain Tree:

$$u^\sharp \to \textsc{Com}_n(v_1^\sharp,\ldots,v_n^\sharp)$$

Chain Trees

$$
\begin{aligned}
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$$\sigma_1(u^\sharp \to \mathrm{Com}_n(v_1^\sharp,\ldots,v_n^\sharp))$$

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$$\mathsf{if}^\sharp(\mathsf{true},x,y) \to \mathrm{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y),\mathsf{p}^\sharp(x)) \qquad \mathsf{p}^\sharp(\mathsf{s}(n)) \to \mathrm{Com}_0$$

$$\mathsf{if}^\sharp(\mathsf{false},x,y) \to \mathrm{Com}_0 \qquad \mathsf{gt}^\sharp(0,k) \to \mathrm{Com}_0$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n),0) \to \mathrm{Com}_0$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) \to \mathrm{Com}_1(\mathsf{gt}^\sharp(n,k))$$

$(\mathcal{D},\mathcal{R})$**-Chain Tree:**

Edge $\sigma_1(u^\sharp \to \mathrm{Com}_n(v_1^\sharp,\dots,v_n^\sharp))$ to $\sigma_2(w^\sharp \to \mathrm{Com}_m(\dots))$ if $v_i^\sharp\sigma_1 \xrightarrow{\mathsf{i}}{}^*_{\mathcal{R}} w^\sharp\sigma_2$

Chain Trees

$$\mathsf{DT}(\mathcal{R}): \quad \mathsf{m}^\sharp(x,y) \to \mathrm{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x,y),x,y),\mathsf{gt}^\sharp(x,y)) \qquad \mathsf{p}^\sharp(0) \to \mathrm{Com}_0$$

$$\mathsf{if}^\sharp(\mathsf{true},x,y) \to \mathrm{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y),\mathsf{p}^\sharp(x)) \qquad \mathsf{p}^\sharp(\mathsf{s}(n)) \to \mathrm{Com}_0$$

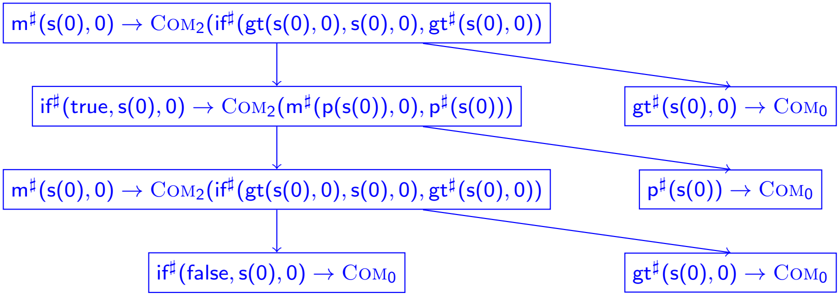
$$\mathsf{if}^\sharp(\mathsf{false},x,y) \to \mathrm{Com}_0 \qquad \mathsf{gt}^\sharp(0,k) \to \mathrm{Com}_0$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n),0) \to \mathrm{Com}_0$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) \to \mathrm{Com}_1(\mathsf{gt}^\sharp(n,k))$$

$(\mathcal{D},\mathcal{R})$-Chain Tree:

Edge $\sigma_1(u^\sharp \to \mathrm{Com}_n(v_1^\sharp,\ldots,v_n^\sharp))$ to $\sigma_2(w^\sharp \to \mathrm{Com}_m(\ldots))$ if $v_i^\sharp\sigma_1 \xrightarrow{\mathsf{i}}{}^*_{\mathcal{R}} w^\sharp\sigma_2$

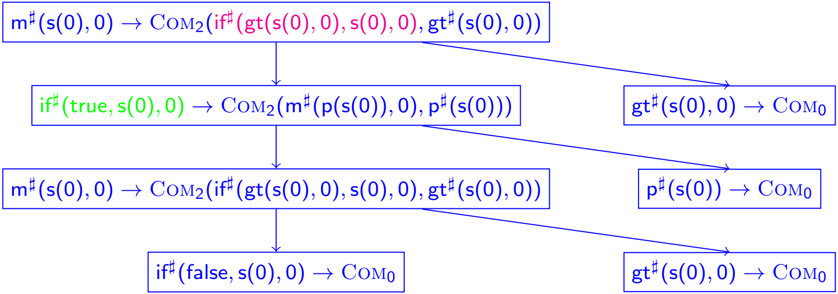


Chain Trees

$$\mathrm{DT}(\mathcal{R}): \quad \begin{aligned} \mathsf{m}^\sharp(x,y) &\to \mathrm{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x,y),x,y),\mathsf{gt}^\sharp(x,y)) \\ \mathsf{if}^\sharp(\mathsf{true},x,y) &\to \mathrm{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y),\mathsf{p}^\sharp(x)) \\ \mathsf{if}^\sharp(\mathsf{false},x,y) &\to \mathrm{Com}_0 \end{aligned} \qquad \begin{aligned} \mathsf{p}^\sharp(0) &\to \mathrm{Com}_0 \\ \mathsf{p}^\sharp(\mathsf{s}(n)) &\to \mathrm{Com}_0 \\ \mathsf{gt}^\sharp(0,k) &\to \mathrm{Com}_0 \\ \mathsf{gt}^\sharp(\mathsf{s}(n),0) &\to \mathrm{Com}_0 \\ \mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) &\to \mathrm{Com}_1(\mathsf{gt}^\sharp(n,k)) \end{aligned}$$

$(\mathcal{D},\mathcal{R})$-Chain Tree:

Edge $\sigma_1(u^\sharp \to \mathrm{Com}_n(v_1^\sharp,\dots,v_n^\sharp))$ to $\sigma_2(w^\sharp \to \mathrm{Com}_m(\dots))$ if $v_i^\sharp\sigma_1 \xrightarrow{\mathrm{i}}{}^*_{\mathcal{R}} w^\sharp\sigma_2$

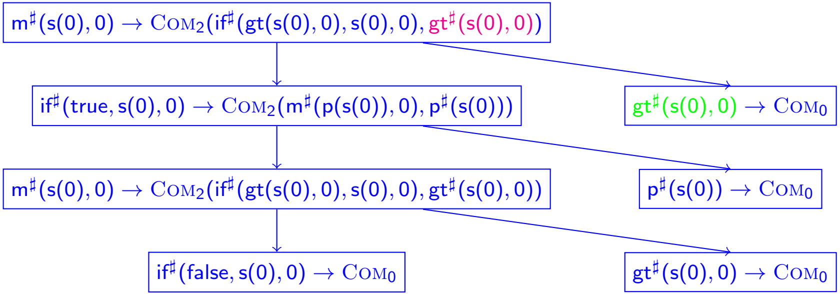


Chain Trees

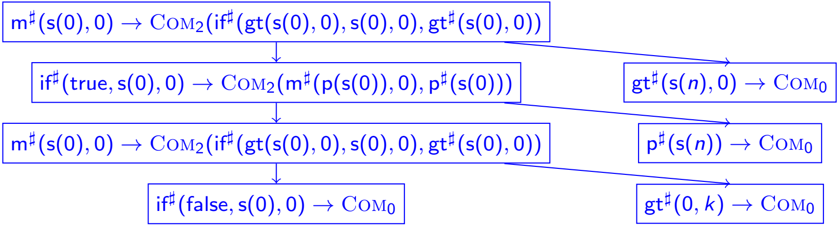
$$
\begin{aligned}
DT(\mathcal{R}):\quad & \mathsf{m}^\sharp(x,y) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(x,y),x,y), \mathsf{gt}^\sharp(x,y)) & \mathsf{p}^\sharp(0) &\to \mathrm{COM}_0\\
& \mathsf{if}^\sharp(\mathsf{true},x,y) \to \mathrm{COM}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y), \mathsf{p}^\sharp(x)) & \mathsf{p}^\sharp(\mathsf{s}(n)) &\to \mathrm{COM}_0\\
& \mathsf{if}^\sharp(\mathsf{false},x,y) \to \mathrm{COM}_0 & \mathsf{gt}^\sharp(0,k) &\to \mathrm{COM}_0\\
& & \mathsf{gt}^\sharp(\mathsf{s}(n),0) &\to \mathrm{COM}_0\\
& & \mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) &\to \mathrm{COM}_1(\mathsf{gt}^\sharp(n,k))
\end{aligned}
$$

$(\mathcal{D},\mathcal{R})$-Chain Tree:

Edge $\sigma_1(u^\sharp \to \mathrm{COM}_n(v_1^\sharp,\ldots,v_n^\sharp))$ to $\sigma_2(w^\sharp \to \mathrm{COM}_m(\ldots))$ if $v_i^\sharp\sigma_1 \xrightarrow{\mathsf{i}}{}^*_{\mathcal{R}} w^\sharp\sigma_2$

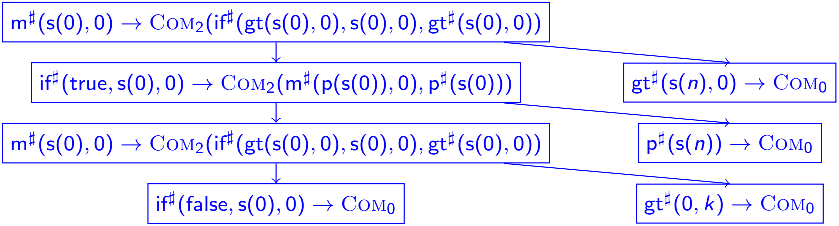


Chain Trees and Complexity



$\iota_{\mathcal{R}}$: length of longest $\xrightarrow{i}_{\mathcal{R}}$-sequence for $|t| \leq n$

Chain Trees and Complexity



$\iota_{\mathcal{R}}$: length of longest $\xrightarrow{\text{i}}_{\mathcal{R}}$-sequence for $|t| \leq n$

$\iota_{\langle \mathcal{D}, \mathcal{R} \rangle}$: maximal number of nodes in chain tree with root $t^\sharp \to \textsc{Com}(\ldots)$ for $|t| \leq n$

Chain Trees and Complexity

$$\mathsf{m}^\sharp(\mathsf{s}(0), 0) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(\mathsf{s}(0), 0), \mathsf{s}(0), 0), \mathsf{gt}^\sharp(\mathsf{s}(0), 0))$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n), 0) \to \textsc{Com}_0$$

$$\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(0), 0) \to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(\mathsf{s}(0)), 0), \mathsf{p}^\sharp(\mathsf{s}(0)))$$

$$\mathsf{p}^\sharp(\mathsf{s}(n)) \to \textsc{Com}_0$$

$$\mathsf{m}^\sharp(\mathsf{s}(0), 0) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(\mathsf{s}(0), 0), \mathsf{s}(0), 0), \mathsf{gt}^\sharp(\mathsf{s}(0), 0))$$

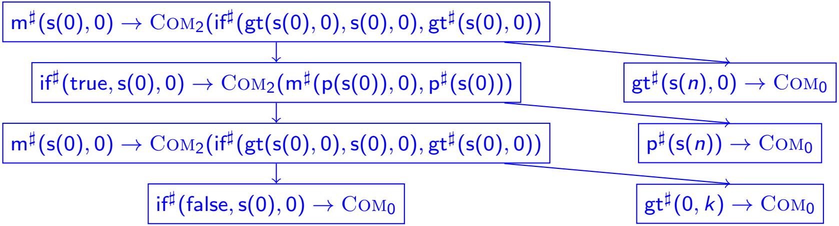
$$\mathsf{gt}^\sharp(0, k) \to \textsc{Com}_0$$

$$\mathsf{if}^\sharp(\mathsf{false}, \mathsf{s}(0), 0) \to \textsc{Com}_0$$

$\iota_{\mathcal{R}}$: length of longest $\xrightarrow{\mathrm{i}}_{\mathcal{R}}$-sequence for $|t| \leq n$

$\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$: maximal number of nodes from $\mathcal{S}$
in chain tree with root $t^\sharp \to \textsc{Com}(\ldots)$ for $|t| \leq n$

Chain Trees and Complexity



$\iota_{\mathcal{R}}$: length of longest $\xrightarrow{\mathrm{i}}_{\mathcal{R}}$-sequence for $|t| \leq n$

$\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle}$: maximal number of nodes from $\mathcal{S}$ in chain tree with root $t^\sharp \to \textsc{Com}(\ldots)$ for $|t| \leq n$

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D},\mathcal{D},\mathcal{R}\rangle}$.

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

⇒Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

⇒Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

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$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

$$P \overset{c}{\leadsto} P'$$

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ where $\iota_P \leq \max(c, \iota_{P'})$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1$ $\quad P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2$ $\quad P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D},\mathcal{D},\mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}',\mathcal{S}',\mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ where $\iota_P \leq \max(c, \iota_{P'})$ P, P' DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2 \overset{c_3}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$

$P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical

$P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k \rangle$ solved

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D},\mathcal{D},\mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}',\mathcal{S}',\mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2 \overset{c_3}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$
$\iota_{P_0} \leq \max(c_1, \iota_{P_1})$

$P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical

$P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k \rangle$ solved

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

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DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2 \overset{c_3}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$
$\iota_{P_0} \leq \max(c_1, c_2, \iota_{P_2})$

$P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical
$P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k \rangle$ solved

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}', \mathcal{S}', \mathcal{R}' \rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2 \overset{c_3}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$ $\quad P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical
$\iota_{P_0} \leq \max(c_1, c_2, \ldots, c_k, \iota_{P_k})$ $\quad P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k \rangle$ solved

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D},\mathcal{D},\mathcal{R} \rangle}$.

$\Rightarrow$Find out $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle}$

$\Rightarrow$Repeatedly replace DT problem $\langle \mathcal{D},\mathcal{S},\mathcal{R} \rangle$ by *simpler* $\langle \mathcal{D}',\mathcal{S}',\mathcal{R}' \rangle$, examine $\iota_{\langle \mathcal{D}',\mathcal{S}',\mathcal{R}' \rangle}$

$\Rightarrow$Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$

DT Processor: $Proc(P) = (c, P')$ where $\iota_P \leq \max(c, \iota_{P'})$ P, P' DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} P_1 \overset{c_2}{\rightsquigarrow} P_2 \overset{c_3}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$ $P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R} \rangle$ canonical

$\iota_{P_0} \leq \max(c_1, c_2, \ldots, c_k)$ $P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k \rangle$ solved

Chain Trees and Complexity

Theorem

If $\mathcal{D} = DT(\mathcal{R})$, then $\iota_{\mathcal{R}} \leq \iota_{\langle \mathcal{D},\mathcal{D},\mathcal{R}\rangle}$.

$\Rightarrow$ Find out $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle}$

$\Rightarrow$ Repeatedly replace DT problem $\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle$ by *simpler* $\langle \mathcal{D}',\mathcal{S}',\mathcal{R}'\rangle$, examine $\iota_{\langle \mathcal{D}',\mathcal{S}',\mathcal{R}'\rangle}$

$\Rightarrow$ Start with canonical DT problem $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R}\rangle$

DT Processor: $Proc(P) = (c, P')$ $\quad P, P'$ DT problems, $c \in \{\mathcal{P}ol_0, \mathcal{P}ol_1, \ldots\}$
where $\iota_P \leq \max(c, \iota_{P'})$

Proof Chain: $P_0 \overset{c_1}{\leadsto} P_1 \overset{c_2}{\leadsto} P_2 \overset{c_3}{\leadsto} \ldots \overset{c_k}{\leadsto} P_k$ $\qquad P_0 = \langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \mathcal{R}\rangle$ canonical
$\iota_{\mathcal{R}} \leq \iota_{P_0} \leq \max(c_1, c_2, \ldots, c_k)$ $\qquad P_k = \langle \mathcal{D}_k, \varnothing, \mathcal{R}_k\rangle$ solved

Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph

Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
edge from $\sigma_1(u \to v)$ to $\sigma_2(w \to t)$ in chain tree

Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
edge from $\sigma_1(u \to v)$ to $\sigma_2(w \to t)$ in chain tree

Example:

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
edge from $\sigma_1(u \to v)$ to $\sigma_2(w \to t)$ in chain tree

Example:

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}$: $\mathsf{q}^\sharp(0, \mathsf{s}(y), \mathsf{s}(z)) \to \mathrm{C}\textsc{om}_0$ $\quad$ $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{C}\textsc{om}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{C}\textsc{om}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

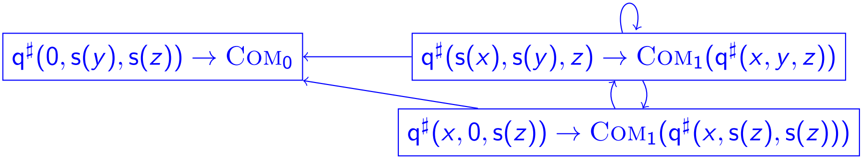
Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
edge from $\sigma_1(u \to v)$ to $\sigma_2(w \to t)$ in chain tree

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}$:



Leaf Removal Processor

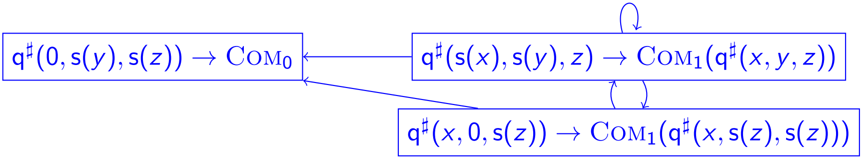
Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
edge from $\sigma_1(u \to v)$ to $\sigma_2(w \to t)$ in chain tree

Leaf Removal Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D} \setminus \{w \to t\},\ \mathcal{S} \setminus \{w \to t\},\ \mathcal{R} \rangle$
if $w \to t$ is leaf in dependency graph

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}$:



Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
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Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle$

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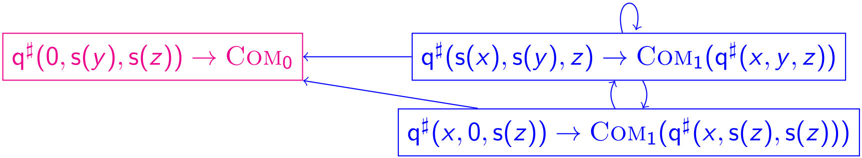
$\mathcal{D}$:



Leaf Removal Processor

Dependency Graph: edge from DT $u \to v$ to $w \to t$ in dep. graph iff
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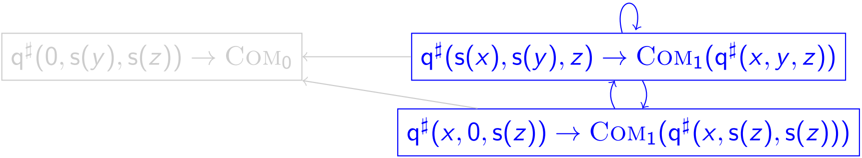
Leaf Removal Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D} \setminus \{w \to t\},\ \mathcal{S} \setminus \{w \to t\},\ \mathcal{R} \rangle$
if $w \to t$ is leaf in dependency graph

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}'$:



Usable Rules Processor

Usable Rules $\mathcal{U}_{\mathcal{R}}(\mathcal{D})$: rules from $\mathcal{R}$ that can reduce rhs of $\mathcal{D}$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Usable Rules Processor

Usable Rules $\mathcal{U}_{\mathcal{R}}(\mathcal{D})$: rules from $\mathcal{R}$ that can reduce rhs of $\mathcal{D}$

Usable Rules Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S},\ \mathcal{U}_{\mathcal{R}}(\mathcal{D}) \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

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Usable Rules Processor

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Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{U}_\mathcal{R}(\mathcal{D}') \rangle$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{Com}_1(\mathsf{q}^\sharp(x, y, z))$

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Usable Rules Processor

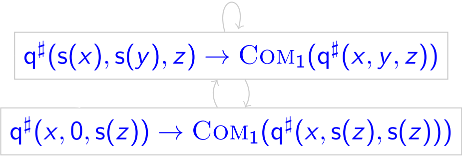
Usable Rules $\mathcal{U}_{\mathcal{R}}(\mathcal{D})$: rules from $\mathcal{R}$ that can reduce rhs of $\mathcal{D}$

Usable Rules Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D},\ \mathcal{S},\ \mathcal{U}_{\mathcal{R}}(\mathcal{D}) \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{D}', \mathcal{U}_{\mathcal{R}}(\mathcal{D}') \rangle$

$\mathcal{U}_{\mathcal{R}}(\mathcal{D}')$:

$\mathcal{D}'$:



Usable Rules Processor

Usable Rules $\mathcal{U}_{\mathcal{R}}(\mathcal{D})$: rules from $\mathcal{R}$ that can reduce rhs of $\mathcal{D}$

Usable Rules Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S},\ \mathcal{U}_{\mathcal{R}}(\mathcal{D}) \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{U}_{\mathcal{R}}(\mathcal{D}')$:

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Extended DT Problems

DT Problem: $\langle \mathcal{D}, \mathcal{S}, \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \text{COM}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \text{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

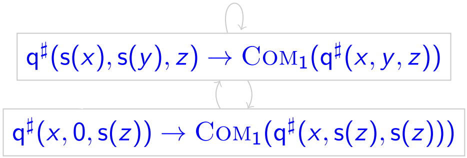
Extended DT Problems

DT Problem: $\langle \mathcal{D}, \mathcal{S}, \quad \mathcal{R} \rangle$

Canonical **DT Problem:** $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

DT Problem: $\langle \mathcal{D}, \mathcal{S}, \quad \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account

Canonical **DT Problem:** $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Extended DT Problems

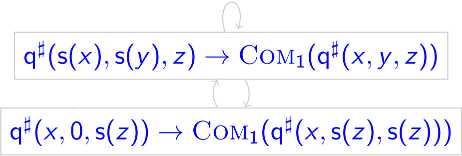
Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account

Canonical DT Problem: $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

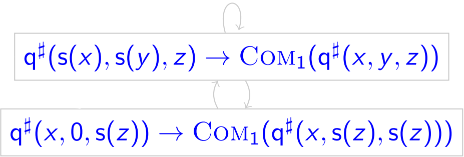
Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account
- $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle} = \begin{cases} \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} > \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \\ & \end{cases}$

Canonical DT Problem: $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \qquad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

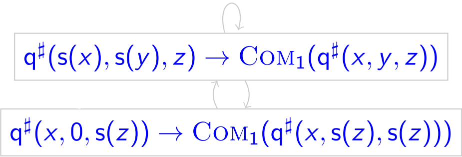
Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account
- $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle} = \begin{cases} \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} > \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \\ \mathcal{P}ol_0, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \end{cases}$

Canonical DT Problem: $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \quad \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \qquad \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account
- $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle} = \begin{cases} \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} > \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \\ \mathcal{P}ol_0, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \end{cases}$

Canonical Extended DT Problem: $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \varnothing, \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, \mathsf{0}, \mathsf{s}(z)) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Extended DT Problems

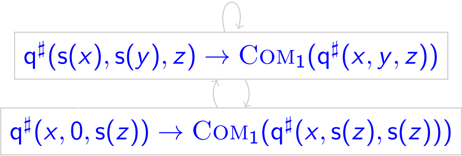
Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle}$, we already took $\iota_{\langle \mathcal{D},\mathcal{K},\mathcal{R}\rangle}$ into account
- $\iota_{\langle \mathcal{D},\mathcal{S},\mathcal{K},\mathcal{R}\rangle} = \begin{cases} \iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle}, & \text{if } \iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle} > \iota_{\langle \mathcal{D},\mathcal{K},\mathcal{R}\rangle} \\ \mathcal{P}ol_0, & \text{if } \iota_{\langle \mathcal{D},\mathcal{S},\mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D},\mathcal{K},\mathcal{R}\rangle} \end{cases}$

Canonical Extended DT Problem: $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \varnothing, \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \mathcal{R} \rangle \qquad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

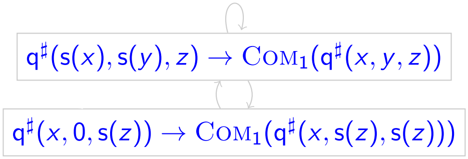
Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account
- $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle} = \begin{cases} \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} > \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \\ \mathcal{P}ol_0, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \end{cases}$

Canonical Extended DT Problem: $\langle \mathrm{DT}(\mathcal{R}), \mathrm{DT}(\mathcal{R}), \varnothing, \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \qquad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing \rangle$

$\mathcal{D}'$:



Extended DT Problems

Extended DT Problem: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$

- when computing $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$, we already took $\iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$ into account
- $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle} = \begin{cases} \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} > \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \\ \mathcal{P}ol_0, & \text{if } \iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle} \end{cases}$

Canonical Extended DT Problem: $\langle \mathsf{DT}(\mathcal{R}), \mathsf{DT}(\mathcal{R}), \varnothing, \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\mathcal{D}'$:

$\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

$\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \succsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

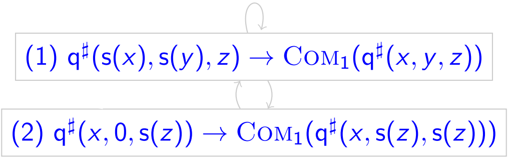
Reduction Pair Processor

Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \succsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_{\succ},\ \mathcal{K} \cup \mathcal{D}_{\succ},\ \mathcal{R} \rangle$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$



Reduction Pair Processor

Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \succsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_\succ,\ \mathcal{K} \cup \mathcal{D}_\succ,\ \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \succsim \cup \succ$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

Termination: $\ell \gtrsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \gtrsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_{\succ},\ \mathcal{K} \cup \mathcal{D}_{\succ},\ \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \gtrsim \cup \succ, \quad \mathcal{R} \subseteq \gtrsim$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \qquad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

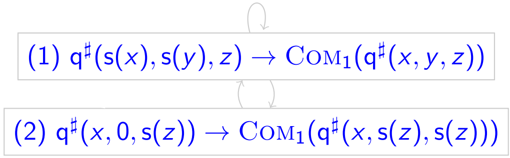
Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \succsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_{\succ},\ \mathcal{K} \cup \mathcal{D}_{\succ},\ \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \succsim \cup \succ, \quad \mathcal{R} \subseteq \succsim$
- m is the maximal degree of polynomials $[f^\sharp]$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$



Reduction Pair Processor

Termination: $\ell \gtrsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \gtrsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D}, \mathcal{S} \setminus \mathcal{D}_{\succ}, \mathcal{K} \cup \mathcal{D}_{\succ}, \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \gtrsim \cup \succ, \quad \mathcal{R} \subseteq \gtrsim$
- m is the maximal degree of polynomials $[f^\sharp]$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

Polynomial Order

$$[\mathrm{Com}_1](x) = x$$
$$[\mathsf{q}^\sharp](x, y, z) = x$$
$$[\mathsf{s}](x) = x + 1$$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

Termination: $\ell \gtrsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \gtrsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_\succ,\ \mathcal{K} \cup \mathcal{D}_\succ,\ \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \gtrsim \cup \succ, \quad \mathcal{R} \subseteq \gtrsim$
- m is the maximal degree of polynomials $[f^\sharp]$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

Polynomial Order

$[\mathrm{Com}_1](x) = x$

$[\mathrm{q}^\sharp](x, y, z) = x$

$[\mathrm{s}](x) = x + 1$

(1) $\mathrm{q}^\sharp(\mathrm{s}(x), \mathrm{s}(y), z) \to \mathrm{Com}_1(\mathrm{q}^\sharp(x, y, z))$

(2) $\mathrm{q}^\sharp(x, 0, \mathrm{s}(z)) \to \mathrm{Com}_1(\mathrm{q}^\sharp(x, \mathrm{s}(z), \mathrm{s}(z)))$

Reduction Pair Processor

Termination: $\ell \succsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \succsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \mathcal{D}_\succ,\ \mathcal{K} \cup \mathcal{D}_\succ,\ \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \succsim \cup \succ, \quad \mathcal{R} \subseteq \succsim$
- m is the maximal degree of polynomials $[f^\sharp]$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

Polynomial Order

$[\mathrm{COM}_1](x) = x$

$[\mathsf{q}^\sharp](x, y, z) = x$

$[\mathsf{s}](x) = x + 1$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \succ \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \succsim \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Reduction Pair Processor

Termination: $\ell \gtrsim r$ for all DPs and rules, remove DPs with $\ell \succ r$

Complexity: $\ell \gtrsim r$ for all DTs and rules, move DTs with $\ell \succ r$ *from* $\mathcal{S}$ *to* $\mathcal{K}$

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D}, \mathcal{S} \setminus \mathcal{D}_{\succ}, \mathcal{K} \cup \mathcal{D}_{\succ}, \mathcal{R} \rangle$ if

- $\mathcal{D} \subseteq \gtrsim \cup \succ, \quad \mathcal{R} \subseteq \gtrsim$
- m is the maximal degree of polynomials $[f^\sharp]$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle$

Polynomial Order

$[\mathrm{COM}_1](x) = x$

$[\mathsf{q}^\sharp](x, y, z) = x$

$[\mathsf{s}](x) = x + 1$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \succ \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \gtrsim \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

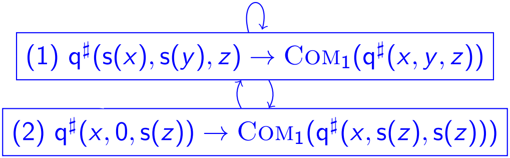
Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R} \rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle$



Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R}\rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing\rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing\rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R}\rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing\rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing\rangle$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathrm{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \mathrm{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

$Pre(\,(2)\,) = \{(1)\}$

Knowledge Propagation Processor

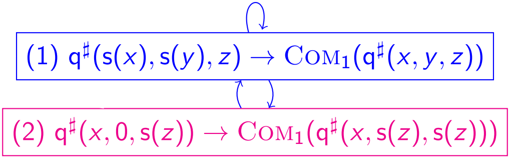
Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R} \rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph
- $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle$: do not take $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle}$ into account if $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R} \rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R} \rangle}$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle$

$Pre(\ (2)\ ) = \{(1)\}$



Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R}\rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph
- $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle$: do not take $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle}$ into account if $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R}\rangle}$

KP Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}, \ \mathcal{S} \setminus \{w \to t\}, \ \mathcal{K} \qquad , \ \mathcal{R}\rangle$
if $w \to t \in \mathcal{S}$ and $Pre(w \to t) \subseteq \mathcal{K}$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R}\rangle \qquad \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing\rangle$
$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing\rangle$

$(1) \ \mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \text{COM}_1(\mathsf{q}^\sharp(x, y, z))$

$Pre(\ (2)\ ) = \{(1)\}$

$(2) \ \mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \text{COM}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R}\rangle}$

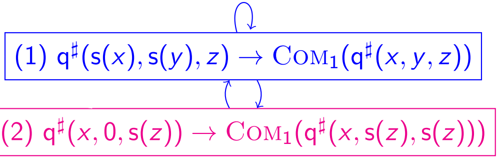
- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph
- $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle$: do not take $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle}$ into account if $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R}\rangle}$

KP Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \{w \to t\},\ \mathcal{K} \cup \{w \to t\},\ \mathcal{R}\rangle$
if $w \to t \in \mathcal{S}$ and $Pre(w \to t) \subseteq \mathcal{K}$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing\rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing\rangle$

$Pre(\ (2)\ ) = \{(1)\}$



Knowledge Propagation Processor

Lemma: $\iota_{\langle \mathcal{D}, \{w \to t\}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, Pre(w \to t), \mathcal{R}\rangle}$

- $Pre(w \to t)$: all predecessors of $w \to t$ in dependency graph
- $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle$: do not take $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle}$ into account if $\iota_{\langle \mathcal{D}, \mathcal{S}, \mathcal{R}\rangle} \leq \iota_{\langle \mathcal{D}, \mathcal{K}, \mathcal{R}\rangle}$

KP Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D},\ \mathcal{S} \setminus \{w \to t\},\ \mathcal{K} \cup \{w \to t\},\ \mathcal{R}\rangle$
if $w \to t \in \mathcal{S}$ and $Pre(w \to t) \subseteq \mathcal{K}$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R}\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing\rangle$
$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing\rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing\rangle$

$Pre(\ (2)\ ) = \{(1)\}$

(1) $\mathsf{q}^\sharp(\mathsf{s}(x), \mathsf{s}(y), z) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, y, z))$

(2) $\mathsf{q}^\sharp(x, 0, \mathsf{s}(z)) \to \textsc{Com}_1(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$ $\iota_{P_0} \leq \max(c_1, \ldots, c_k)$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$ $\quad \iota_{\mathcal{R}} \leq \iota_{P_0} \leq \max(c_1, \ldots, c_k)$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \dots \overset{c_k}{\rightsquigarrow} P_k$ $\qquad \iota_{\mathcal{R}} \leq \iota_{P_0} \leq \max(c_1, \dots, c_k)$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \dots \overset{c_k}{\rightsquigarrow} P_k$ $\qquad \iota_{\mathcal{R}} \leq \iota_{P_0} \leq \max(c_1, \dots, c_k)$

$\mathcal{R}$: $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0$, $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z)$, $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$
$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

$$\iota_{\mathcal{R}} \leq \max(\mathcal{P}ol_0, \mathcal{P}ol_0, \mathcal{P}ol_1, \mathcal{P}ol_0)$$

Knowledge Propagation Processor

Proof Chain: $P_0 \overset{c_1}{\rightsquigarrow} \ldots \overset{c_k}{\rightsquigarrow} P_k$ $\qquad \iota_{\mathcal{R}} \leq \iota_{P_0} \leq \max(c_1, \ldots, c_k)$

$\mathcal{R}: \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to 0, \;\; \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \mathsf{q}(x, y, z), \;\; \mathsf{q}(x, 0, \mathsf{s}(z)) \to \mathsf{s}(\mathsf{q}(x, \mathsf{s}(z), \mathsf{s}(z)))$

Example: $\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{D}', \varnothing, \varnothing \rangle$

$\overset{\mathcal{P}ol_1}{\rightsquigarrow} \langle \mathcal{D}', \{(2)\}, \{(1)\}, \varnothing \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \varnothing, \{(1), (2)\}, \varnothing \rangle$

$$\iota_{\mathcal{R}} \leq \max(\mathcal{P}ol_0, \mathcal{P}ol_0, \mathcal{P}ol_1, \mathcal{P}ol_0) = \mathcal{P}ol_1$$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

$$
\mathcal{R} : \quad
\begin{array}{rcl}
\mathsf{m}(x,y) & \to & \mathsf{if}(\mathsf{gt}(x,y),x,y) \\
\mathsf{if}(\mathsf{false},x,y) & \to & 0 \\
\mathsf{if}(\mathsf{true},x,y) & \to & \mathsf{s}(\mathsf{m}(\mathsf{p}(x),y))
\end{array}
\quad
\begin{array}{rcl}
\mathsf{gt}(0,k) & \to & \mathsf{false} \\
\mathsf{gt}(\mathsf{s}(n),0) & \to & \mathsf{true} \\
\mathsf{gt}(\mathsf{s}(n),\mathsf{s}(k)) & \to & \mathsf{gt}(n,k)
\end{array}
\quad
\begin{array}{rcl}
\mathsf{p}(0) & \to & 0 \\
\mathsf{p}(\mathsf{s}(n)) & \to & n
\end{array}
$$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle$

$$
\mathcal{R} : \quad
\begin{array}{rcl rcl rcl}
\mathsf{m}(x,y) & \to & \mathsf{if}(\mathsf{gt}(x,y),x,y) & \mathsf{gt}(0,k) & \to & \mathsf{false} & \mathsf{p}(0) & \to & 0 \\
\mathsf{if}(\mathsf{false},x,y) & \to & 0 & \mathsf{gt}(\mathsf{s}(n),0) & \to & \mathsf{true} & \mathsf{p}(\mathsf{s}(n)) & \to & n \\
\mathsf{if}(\mathsf{true},x,y) & \to & \mathsf{s}(\mathsf{m}(\mathsf{p}(x),y)) & \mathsf{gt}(\mathsf{s}(n),\mathsf{s}(k)) & \to & \mathsf{gt}(n,k) & & &
\end{array}
$$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where
in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0{}^*}{\rightsquigarrow} \quad \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle$$

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{gt}(0, k) & \to & \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) & \to & \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) & \to & \mathsf{gt}(n, k) \end{array} \qquad \begin{array}{rcl} \mathsf{p}(0) & \to & 0 \\ \mathsf{p}(\mathsf{s}(n)) & \to & n \end{array}$$

$\mathcal{D}_1$: $\quad \mathsf{m}^\sharp(x, y) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

$$\mathsf{if}^\sharp(\mathsf{true}, x, y) \to \mathrm{COM}_2(\mathsf{m}^\sharp(\mathsf{p}(x), y), \mathsf{p}^\sharp(x)) \qquad \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \mathrm{COM}_1(\mathsf{gt}^\sharp(n, k))$$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

Narrowings of $\mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y),\ \mathsf{gt}^\sharp(x, y))$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0}{\rightsquigarrow}^* \quad \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle$$

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{gt}(0, k) & \to & \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) & \to & \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) & \to & \mathsf{gt}(n, k) \end{array} \qquad \begin{array}{rcl} \mathsf{p}(0) & \to & 0 \\ \mathsf{p}(\mathsf{s}(n)) & \to & n \end{array}$$

$\mathcal{D}_1$: $\quad \mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

$\mathsf{if}^\sharp(\mathsf{true}, x, y) \to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x), y), \mathsf{p}^\sharp(x)) \qquad \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \textsc{Com}_1(\mathsf{gt}^\sharp(n, k))$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

Narrowings of $\mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

- $\mathsf{m}^\sharp(0, k) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{false}, 0, k), \mathsf{gt}^\sharp(0, k))$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0{}^*}{\rightsquigarrow} \quad \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle$$

$\mathcal{R}_1$:

$$\begin{aligned} \mathsf{gt}(0, k) &\to \mathsf{false} & \mathsf{p}(0) &\to 0 \\ \mathsf{gt}(\mathsf{s}(n), 0) &\to \mathsf{true} & \mathsf{p}(\mathsf{s}(n)) &\to n \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\to \mathsf{gt}(n, k) \end{aligned}$$

$\mathcal{D}_1$: $\mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

$\mathsf{if}^\sharp(\mathsf{true}, x, y) \to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x), y), \mathsf{p}^\sharp(x))$ $\qquad$ $\mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \textsc{Com}_1(\mathsf{gt}^\sharp(n, k))$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

Narrowings of $\mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

- $\mathsf{m}^\sharp(0, k) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{false}, 0, k), \mathsf{gt}^\sharp(0, k))$
- $\mathsf{m}^\sharp(\mathsf{s}(n), 0) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), 0), \mathsf{gt}^\sharp(\mathsf{s}(n), 0))$

$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0{}^*}{\leadsto} \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle$

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{gt}(0, k) & \to & \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) & \to & \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) & \to & \mathsf{gt}(n, k) \end{array} \qquad \begin{array}{rcl} \mathsf{p}(0) & \to & 0 \\ \mathsf{p}(\mathsf{s}(n)) & \to & n \end{array}$$

$\mathcal{D}_1$: $\mathsf{m}^\sharp(x, y) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

$\mathsf{if}^\sharp(\mathsf{true}, x, y) \to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x), y), \mathsf{p}^\sharp(x))$ $\qquad$ $\mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \textsc{Com}_1(\mathsf{gt}^\sharp(n, k))$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

Narrowings of $\mathsf{m}^\sharp(x, y) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

- $\mathsf{m}^\sharp(0, k) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{false}, 0, k), \mathsf{gt}^\sharp(0, k))$
- $\mathsf{m}^\sharp(\mathsf{s}(n), 0) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), 0), \mathsf{gt}^\sharp(\mathsf{s}(n), 0))$
- $\mathsf{m}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(n, k), \mathsf{s}(n), \mathsf{s}(k)), \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)))$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle$$

$\mathcal{R}_1$:

$$\begin{aligned} \mathsf{gt}(0, k) &\to \mathsf{false} & \mathsf{p}(0) &\to 0 \\ \mathsf{gt}(\mathsf{s}(n), 0) &\to \mathsf{true} & \mathsf{p}(\mathsf{s}(n)) &\to n \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\to \mathsf{gt}(n, k) \end{aligned}$$

$\mathcal{D}_1$: $\mathsf{m}^\sharp(x, y) \to \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(x, y), x, y), \mathsf{gt}^\sharp(x, y))$

$\mathsf{if}^\sharp(\mathsf{true}, x, y) \to \mathrm{COM}_2(\mathsf{m}^\sharp(\mathsf{p}(x), y), \mathsf{p}^\sharp(x))$ $\quad$ $\mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \mathrm{COM}_1(\mathsf{gt}^\sharp(n, k))$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\leadsto} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

Narrowings of $\mathsf{m}^\sharp(x,y) \ \to\ \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(x,y),x,y),\ \mathsf{gt}^\sharp(x,y))$

- $\mathsf{m}^\sharp(\mathsf{s}(n),0) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{true},\mathsf{s}(n),0),\mathsf{gt}^\sharp(\mathsf{s}(n),0))$
- $\mathsf{m}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(n,k),\mathsf{s}(n),\mathsf{s}(k)),\mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)))$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \quad \overset{\mathcal{P}ol_0}{\leadsto}^{*} \ \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\leadsto}^{*} \ \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle$$

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{gt}(0,k) & \to & \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n),0) & \to & \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n),\mathsf{s}(k)) & \to & \mathsf{gt}(n,k) \end{array} \qquad \begin{array}{rcl} \mathsf{p}(0) & \to & 0 \\ \mathsf{p}(\mathsf{s}(n)) & \to & n \end{array}$$

$\mathcal{D}_2$: $\mathsf{m}^\sharp(\mathsf{s}(n),0) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{true},\mathsf{s}(n),0),\mathsf{gt}^\sharp(\mathsf{s}(n),0))$

$\mathsf{m}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) \to \textsc{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(n,k),\mathsf{s}(n),\mathsf{s}(k)),\mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)))$

$\mathsf{if}^\sharp(\mathsf{true},x,y) \to \textsc{Com}_2(\mathsf{m}^\sharp(\mathsf{p}(x),y),\mathsf{p}^\sharp(x)) \qquad \mathsf{gt}^\sharp(\mathsf{s}(n),\mathsf{s}(k)) \to \textsc{Com}_1(\mathsf{gt}^\sharp(n,k))$

Narrowing Processor

Narrowing Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow} \langle \mathcal{D}', \mathcal{S}', \mathcal{K}', \mathcal{R} \rangle$ where in $\mathcal{D}'$, $\mathcal{S}'$, some $w \to t$ is replaced by all its narrowings

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_3, \mathcal{D}_3, \varnothing, \mathcal{R}_2 \rangle$$

$\mathcal{R}_2$:

$$\begin{aligned} \mathsf{gt}(0, k) &\to \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) &\to \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\to \mathsf{gt}(n, k) \end{aligned}$$

$\mathcal{D}_3$:

$$\begin{aligned} \mathsf{m}^\sharp(\mathsf{s}(n), 0) &\to \mathrm{Com}_2(\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), 0), \mathsf{gt}^\sharp(\mathsf{s}(n), 0)) \\ \mathsf{m}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) &\to \mathrm{Com}_2(\mathsf{if}^\sharp(\mathsf{gt}(n, k), \mathsf{s}(n), \mathsf{s}(k)), \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k))) \\ \mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), y) &\to \mathrm{Com}_2(\mathsf{m}^\sharp(n, y), \mathsf{p}^\sharp(\mathsf{s}(n))) \end{aligned}$$

$$\mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \to \mathrm{Com}_1(\mathsf{gt}^\sharp(n, k))$$

Narrowing Processor

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D}, \mathcal{S} \setminus \mathcal{D}_\succ, \mathcal{K} \cup \mathcal{D}_\succ, \mathcal{R} \rangle$ where m is the maximal degree of polynomials $[f^\sharp]$

Polynomial Order

- $[0] = [\mathsf{true}] = [\mathsf{false}] = [\mathsf{p}^\sharp](x) = 0, \quad [\mathsf{s}](x) = x + 2$
- $[\mathsf{gt}](x, y) = [\mathsf{gt}^\sharp](x, y) = x$
- $[\mathsf{m}^\sharp](x, y) = (x + 1)^2, \quad [\mathsf{if}^\sharp](x, y, z) = y^2$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}^* \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}^* \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}^* \langle \mathcal{D}_3, \mathcal{D}_3, \varnothing, \mathcal{R}_2 \rangle$$

$\mathcal{R}_2$:

$$\begin{aligned} \mathsf{gt}(0, k) &\succsim \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) &\succsim \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\succsim \mathsf{gt}(n, k) \end{aligned}$$

$\mathcal{D}_3$:

$$\begin{aligned} \mathsf{m}^\sharp(\mathsf{s}(n), 0) &\succ \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), 0), \mathsf{gt}^\sharp(\mathsf{s}(n), 0)) \\ \mathsf{m}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) &\succ \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(n, k), \mathsf{s}(n), \mathsf{s}(k)), \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k))) \\ \mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), y) &\succ \mathrm{COM}_2(\mathsf{m}^\sharp(\mathsf{s}(n), y), \mathsf{p}^\sharp(\mathsf{s}(n))) \quad \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \succ \mathrm{COM}_1(\mathsf{gt}^\sharp(n, k)) \end{aligned}$$

Narrowing Processor

Reduction Pair Processor: $\langle \mathcal{D}, \mathcal{S}, \mathcal{K}, \mathcal{R} \rangle \overset{\mathcal{P}ol_m}{\rightsquigarrow} \langle \mathcal{D}, \mathcal{S} \setminus \mathcal{D}_\succ, \mathcal{K} \cup \mathcal{D}_\succ, \mathcal{R} \rangle$ where m is the maximal degree of polynomials $[f^\sharp]$

Polynomial Order

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- $[\mathsf{gt}](x, y) = [\mathsf{gt}^\sharp](x, y) = x$
- $[\mathsf{m}^\sharp](x, y) = (x + 1)^2, \quad [\mathsf{if}^\sharp](x, y, z) = y^2$

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_3, \mathcal{D}_3, \varnothing, \mathcal{R}_2 \rangle$$
$$\overset{\mathcal{P}ol_2}{\rightsquigarrow} \langle \mathcal{D}_3, \varnothing, \mathcal{D}_3, \mathcal{R}_2 \rangle$$

$\mathcal{R}_2$:

$$\begin{aligned} \mathsf{gt}(0, k) &\succsim \mathsf{false} \\ \mathsf{gt}(\mathsf{s}(n), 0) &\succsim \mathsf{true} \\ \mathsf{gt}(\mathsf{s}(n), \mathsf{s}(k)) &\succsim \mathsf{gt}(n, k) \end{aligned}$$

$$\begin{aligned} \mathcal{D}_3: \quad \mathsf{m}^\sharp(\mathsf{s}(n), 0) &\succ \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), 0), \mathsf{gt}^\sharp(\mathsf{s}(n), 0)) \\ \mathsf{m}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) &\succ \mathrm{COM}_2(\mathsf{if}^\sharp(\mathsf{gt}(n, k), \mathsf{s}(n), \mathsf{s}(k)), \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k))) \\ \mathsf{if}^\sharp(\mathsf{true}, \mathsf{s}(n), y) &\succ \mathrm{COM}_2(\mathsf{m}^\sharp(\mathsf{s}(n), y), \mathsf{p}^\sharp(\mathsf{s}(n))) \quad \mathsf{gt}^\sharp(\mathsf{s}(n), \mathsf{s}(k)) \succ \mathrm{COM}_1(\mathsf{gt}^\sharp(n, k)) \end{aligned}$$

Narrowing Processor

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^* \langle \mathcal{D}_3, \mathcal{D}_3, \varnothing, \mathcal{R}_2 \rangle$$

$$\overset{\mathcal{P}ol_2}{\rightsquigarrow} \langle \mathcal{D}_3, \varnothing, \mathcal{D}_3, \mathcal{R}_2 \rangle$$

$$\iota_{\mathcal{R}} \leq \max(\mathcal{P}ol_0, \ldots, \mathcal{P}ol_0, \mathcal{P}ol_2)$$

Narrowing Processor

$$\langle \mathcal{D}, \mathcal{D}, \varnothing, \mathcal{R} \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_1, \mathcal{D}_1, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_2, \mathcal{D}_2, \varnothing, \mathcal{R}_1 \rangle \overset{\mathcal{P}ol_0}{\rightsquigarrow}{}^{*} \langle \mathcal{D}_3, \mathcal{D}_3, \varnothing, \mathcal{R}_2 \rangle$$

$$\overset{\mathcal{P}ol_2}{\rightsquigarrow} \langle \mathcal{D}_3, \varnothing, \mathcal{D}_3, \mathcal{R}_2 \rangle$$

$$\iota_{\mathcal{R}} \leq \max(\mathcal{P}ol_0, \ldots, \mathcal{P}ol_0, \mathcal{P}ol_2) = \mathcal{P}ol_2$$

DT Framework for Innermost Complexity Analysis

- *Direct* adaption of DP framework for termination analysis

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- Experiments on 1323 TRSs from *Termination Problem Data Base*

AProVE:	618 examples with polynomial runtime
CaT:	447 examples with polynomial runtime
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		CaT					
		$\mathcal{P}ol_0$	$\mathcal{P}ol_1$	$\mathcal{P}ol_2$	$\mathcal{P}ol_3$	no result	$\sum$
AProVE	$\mathcal{P}ol_0$	-	182	-	-	27	209
	$\mathcal{P}ol_1$	-	187	7	-	76	270
	$\mathcal{P}ol_2$	-	32	2	-	83	117
	$\mathcal{P}ol_3$	-	6	-	-	16	22
	no result	-	27	3	1	674	705
	$\sum$	0	434	12	1	876	1323

DT Framework for Innermost Complexity Analysis

- *Direct* adaption of DP framework for termination analysis
- *Modular* combination of different techniques
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AProVE: 618 examples with polynomial runtime
CaT: 447 examples with polynomial runtime
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		TCT					
		$\mathcal{P}ol_0$	$\mathcal{P}ol_1$	$\mathcal{P}ol_2$	$\mathcal{P}ol_3$	no result	$\sum$
AProVE	$\mathcal{P}ol_0$	10	157	-	-	42	209
	$\mathcal{P}ol_1$	-	152	1	-	117	270
	$\mathcal{P}ol_2$	-	35	-	-	82	117
	$\mathcal{P}ol_3$	-	5	-	-	17	22
	no result	-	22	3	-	680	705
	$\sum$	10	371	4	0	938	1323

DT Framework for Innermost Complexity Analysis

- *Direct* adaption of DP framework for termination analysis
- *Modular* combination of different techniques
- Experiments on 1323 TRSs from *Termination Problem Data Base*

AProVE:	618 examples with polynomial runtime
CaT:	447 examples with polynomial runtime
TCT:	385 examples with polynomial runtime

		TCT					
		$\mathcal{P}ol_0$	$\mathcal{P}ol_1$	$\mathcal{P}ol_2$	$\mathcal{P}ol_3$	no result	$\sum$
AProVE	$\mathcal{P}ol_0$	10	157	-	-	42	209
	$\mathcal{P}ol_1$	-	152	1	-	117	270
	$\mathcal{P}ol_2$	-	35	-	-	82	117
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	no result	-	22	3	-	680	705
	$\sum$	10	371	4	0	938	1323

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