Termination of Polynomial Loops

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joint work with Marcel Hark and Florian Frohn

Termination and complexity analysis for programs

`Java`

`C`

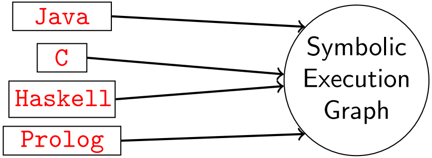
`Haskell`

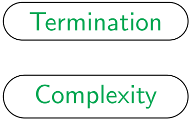
`Prolog`

Termination

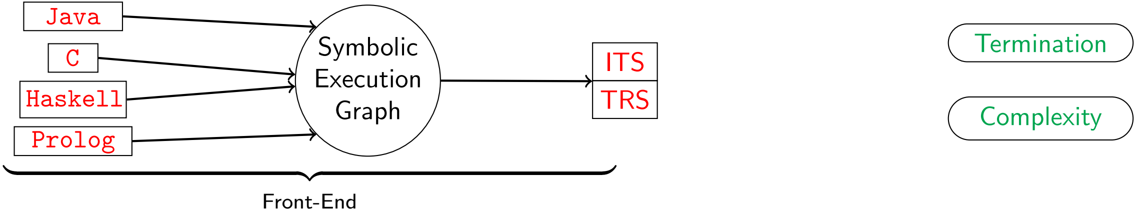
Complexity

Termination and complexity analysis for programs





Termination and complexity analysis for programs



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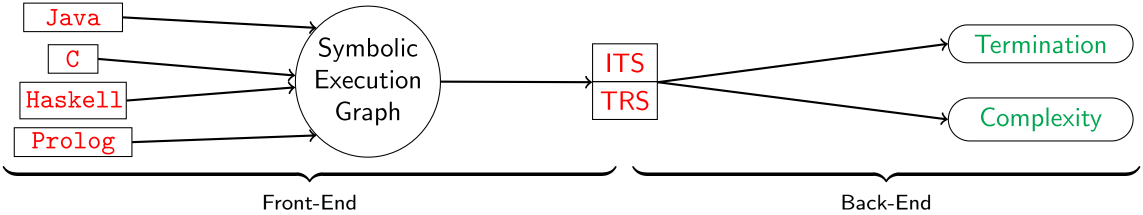

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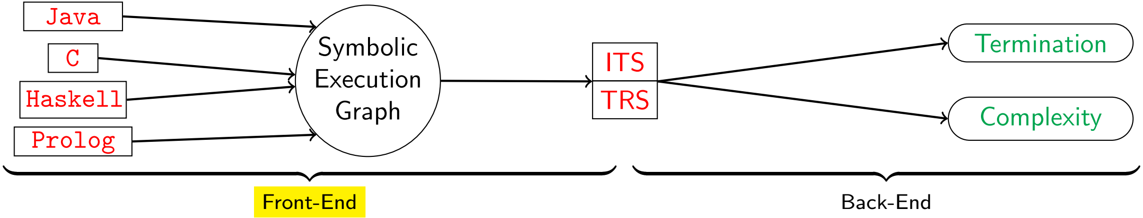

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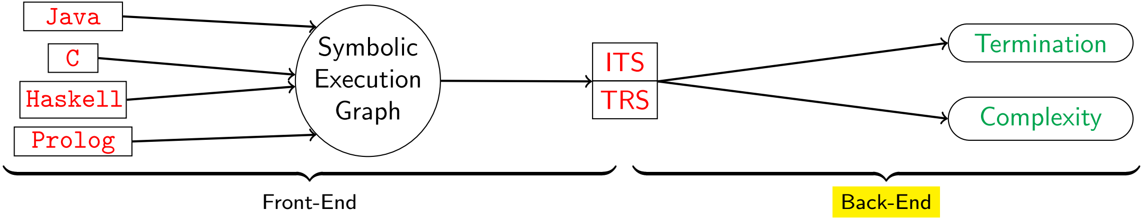

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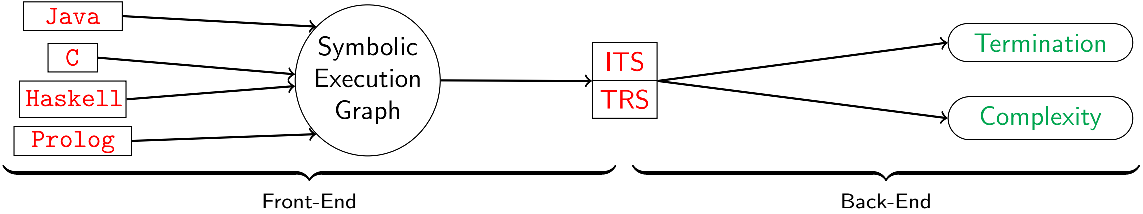

- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems
- powerful termination and complexity analysis implemented in AProVE, KoAT, and LoAT
  - Termination Competition since 2004 (`Java`, `C`, `Haskell`, `Prolog`, `TRS`)
  - SV-COMP since 2014 (`C`)

Programs where termination is decidable

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  - termination decidable over $\mathbb{Z}$ if A is triangular
- Hosseini, Ouaknine, Worrell (ICALP 2019)
  - termination decidable over $\mathbb{Z}$

Termination of $\underline{t}$riangular $\underline{w}$eakly $\underline{n}$on-linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

$$\textbf{while}\ (\ \varphi\ )\ \textbf{do}\ \begin{bmatrix} x_1 \\ \cdots \\ x_d \end{bmatrix} \leftarrow \begin{bmatrix} c_1 \cdot x_1 + p_1 \\ \cdots \\ c_d \cdot x_d + p_d \end{bmatrix}$$

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  and $(b_\ell, a_\ell) >_{lex} \ldots >_{lex} (b_2, a_2) >_{lex} (b_1, a_1)$

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Check $\exists x_1, x_2, x_3 \in \mathcal{S},\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p > 0$

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$$
\begin{array}{rll}
\mathrm{red}(p) = & & \alpha_\ell > 0 \\
& \vee & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} > 0 \\
& \vee & \qquad \ldots \\
& \vee & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} = 0 \ \wedge\ \ldots\ \wedge\ \alpha_2 = 0 \ \wedge\ \alpha_1 > 0
\end{array}
$$

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  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathsf{red}(\varphi(\vec{q}))$ is valid

- Eliminate $\forall$ by examining dominant terms in $\varphi(\vec{q})$'s inequations

  $\Rightarrow$ replace $p > 0$ in $\varphi(\vec{q})$ with $p = \sum_{1 \leq j \leq \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ by

$$\begin{aligned} \mathsf{red}(p) = \quad & \alpha_\ell > 0 \\ \vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} > 0 \\ \vee \quad & \qquad \dots \\ \vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} = 0 \ \wedge\ \dots\ \wedge\ \alpha_2 = 0 \ \wedge\ \alpha_1 > 0 \end{aligned}$$

while $(\ x_1 + x_2^2 > 0\ )$ **do** $(x_1, x_2, x_3) \ \leftarrow\ (x_1 + x_2^2 \cdot x_3,\ x_2 - 2 \cdot x_3^2,\ x_3)$

Check $\exists x_1, x_2, x_3 \in \mathcal{S},\ n_0 \in \mathbb{N}.\ \forall n \in \mathbb{N}_{>n_0}.\ p > 0$ for $p = \alpha_4 \cdot n^3 + \alpha_3 \cdot n^2 + \alpha_2 \cdot n + \alpha_1$ where

$$\alpha_4 = \tfrac{4}{3} \cdot x_3^5 \qquad \alpha_3 = -2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4$$

$$\alpha_2 = x_2^2 \cdot x_3 + \tfrac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2 \qquad \alpha_1 = x_1 + x_2^2$$

Termination of $\underline{t}$riangular $\underline{w}$eakly $\underline{n}$on-linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow \vec{u}$

- Compute normalized closed form $\vec{q} = \vec{u}^n$ for all large enough n

  $\Rightarrow$ $q = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ with polynomials α_j over $\vec{x}$, $a_j \in \mathbb{N}$, $\in \mathcal{S}_{>0}$

  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

- Eliminate $\forall$ by examining dominant terms in $\varphi(\vec{q})$'s inequations

  $\Rightarrow$ replace $p > 0$ in $\varphi(\vec{q})$ with $p = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ by

$$\begin{aligned} \mathrm{red}(p) = \quad & \alpha_\ell > 0 \\ \vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} > 0 \\ \vee \quad & \ldots \\ \vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} = 0 \ \wedge\ \ldots \ \wedge\ \alpha_2 = 0 \ \wedge\ \alpha_1 > 0 \end{aligned}$$

while $(\ x_1 + x_2^2 > 0\ )$ **do** $(x_1, x_2, x_3) \leftarrow (x_1 + x_2^2 \cdot x_3,\ x_2 - 2 \cdot x_3^2,\ x_3)$

Check $\exists x_1, x_2, x_3 \in \mathcal{S}.\ \mathrm{red}(p)$ for $p = \alpha_4 \cdot n^3 + \alpha_3 \cdot n^2 + \alpha_2 \cdot n + \alpha_1$ where

$$\alpha_4 = \tfrac{4}{3} \cdot x_3^5 \qquad \alpha_3 = -2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4$$

$$\alpha_2 = x_2^2 \cdot x_3 + \tfrac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2 \qquad \alpha_1 = x_1 + x_2^2$$

Termination of triangular weakly non-linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow \vec{u}$

- Compute normalized closed form $\vec{q} = \vec{u}^n$ for all large enough n

  $\Rightarrow$ $q = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ with polynomials α_j over $\vec{x}$, $a_j \in \mathbb{N}$, $\in \mathcal{S}_{>0}$

  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

- Eliminate $\forall$ by examining dominant terms in $\varphi(\vec{q})$'s inequations

  $\Rightarrow$ replace $p > 0$ in $\varphi(\vec{q})$ with $p = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ by

$$
\begin{array}{rrl}
\mathrm{red}(p) = & & \alpha_\ell > 0 \\
 & \vee & \alpha_\ell = 0 \ \wedge \ \alpha_{\ell-1} > 0 \\
 & \vee & \qquad \dots \\
 & \vee & \alpha_\ell = 0 \ \wedge \ \alpha_{\ell-1} = 0 \ \wedge \ \dots \ \wedge \ \alpha_2 = 0 \ \wedge \ \alpha_1 > 0
\end{array}
$$

while $(\ x_1 + x_2^2 > 0\ )$ **do** $(x_1, x_2, x_3) \ \leftarrow \ (x_1 + x_2^2 \cdot x_3,\ x_2 - 2 \cdot x_3^2,\ x_3)$ **non-terminating**

Check $\exists x_1, x_2, x_3 \in \mathcal{S}.\ \mathrm{red}(p)$ for $p = \alpha_4 \cdot n^3 + \alpha_3 \cdot n^2 + \alpha_2 \cdot n + \alpha_1$ where

$$\alpha_4 = \tfrac{4}{3} \cdot x_3^5 \qquad \alpha_3 = -2 \cdot x_3^5 - 2 \cdot x_2 \cdot x_3^3 + 4 \cdot x_3^4$$

$$\alpha_2 = x_2^2 \cdot x_3 + \tfrac{2}{3} \cdot x_3^5 + 2 \cdot x_2 \cdot x_3^3 - 4 \cdot x_2 \cdot x_3^2 \qquad \alpha_1 = x_1 + x_2^2$$

Termination of triangular weakly non-linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow \vec{u}$

- Compute normalized closed form $\vec{q} = \vec{u}^n$ for all large enough n

  $\Rightarrow$ $q = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ with polynomials α_j over $\vec{x}$, $a_j \in \mathbb{N}$, $\in \mathcal{S}_{>0}$

  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

- Eliminate $\forall$ by examining dominant terms in $\varphi(\vec{q})$'s inequations

  $\Rightarrow$ replace $p > 0$ in $\varphi(\vec{q})$ with $p = \sum_{1 \le j \le \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ by

$$
\begin{aligned}
\mathrm{red}(p) = \quad & \alpha_\ell > 0 \\
\vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} > 0 \\
\vee \quad & \qquad \ldots \\
\vee \quad & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} = 0 \ \wedge\ \ldots \ \wedge\ \alpha_2 = 0 \ \wedge\ \alpha_1 > 0
\end{aligned}
$$

Theorem

- Termination of twn-loops *decidable* for $\mathcal{S} \in \{\mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$

Termination of <u>t</u>riangular <u>w</u>eakly <u>n</u>on-linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow \vec{u}$

- Compute normalized closed form $\vec{q} = \vec{u}^n$ for all large enough n

  $\Rightarrow q = \sum_{1 \leq j \leq \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ with polynomials α_j over $\vec{x}$, $a_j \in \mathbb{N}$, $\in \mathcal{S}_{>0}$

  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

- Eliminate $\forall$ by examining dominant terms in $\varphi(\vec{q})$'s inequations

  $\Rightarrow$ replace $p > 0$ in $\varphi(\vec{q})$ with $p = \sum_{1 \leq j \leq \ell} \alpha_j \cdot n^{a_j} \cdot b_j{}^n$ by

$$
\begin{array}{rcl}
\mathrm{red}(p) = & & \alpha_\ell > 0 \\
& \vee & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} > 0 \\
& \vee & \dots \\
& \vee & \alpha_\ell = 0 \ \wedge\ \alpha_{\ell-1} = 0 \ \wedge\ \dots\ \wedge\ \alpha_2 = 0 \ \wedge\ \alpha_1 > 0
\end{array}
$$

Theorem

- Termination of twn-loops *decidable* for $\mathcal{S} \in \{\mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$
- Non-termination of twn-loops *semi-decidable* for $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}\}$

Termination of linear loops over $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\, \varphi \,)$ **do** $\vec{x} \leftarrow A \cdot \vec{x} + \vec{c}$

Termination of linear loops over $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

Termination of linear loops over $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form

Termination of linear loops over $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

- If A is not triangular, transform it into Jordan normal form

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

$$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} x_1 + n \cdot x_2 \\ x_2 \\ x_3 \cdot 2^n + \left(\frac{x_4}{2} - \frac{x_5}{8}\right) \cdot n \cdot 2^n + \frac{x_5}{8} \cdot n^2 \cdot 2^n \\ x_4 \cdot 2^n + \frac{x_5}{2} \cdot n \cdot 2^n \\ x_5 \cdot 2^n \end{bmatrix}$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

- If A is not triangular, transform it into Jordan normal form

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n
  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} x_1 + n \cdot x_2 \\ x_2 \\ x_3 \cdot 2^n + (\frac{x_4}{2} - \frac{x_5}{8}) \cdot n \cdot 2^n + \frac{x_5}{8} \cdot n^2 \cdot 2^n \\ x_4 \cdot 2^n + \frac{x_5}{2} \cdot n \cdot 2^n \\ x_5 \cdot 2^n \end{bmatrix}$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n
  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

$\varphi:\ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}):\ p > 0$ with

$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_\mathbb{A}\}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n
  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

Theorem

- Termination of linear loops with rational eigenvalues is *Co-NP complete* for $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}\}$

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \ p > 0$ with

$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n
  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \mathrm{red}(\varphi(\vec{q}))$ is valid

Theorem

- Termination of linear loops with rational eigenvalues is *Co-NP complete* for $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}\}$
- Termination of linear-update loops with real eigenvalues is $\forall\mathbb{R}$ *complete* for $\mathcal{S} \in \{\mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \ p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- Compute normalized closed form $\vec{q} = A^n \cdot \vec{x}$ for all large enough n
  $\Rightarrow$ Loop is *non-terminating* iff $\exists \vec{x} \in \mathcal{S}^d.\ \text{red}(\varphi(\vec{q}))$ is valid

Theorem

- Termination of linear loops with rational eigenvalues is *Co-NP complete* for $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}\}$
- Termination of linear-update loops with real eigenvalues is $\forall\mathbb{R}$ *complete* for $\mathcal{S} \in \{\mathbb{R}_{\mathbb{A}}, \mathbb{R}\}$

Proof of Co-NP hardness: ξ unsatisfiable iff
while $(\xi\,[\,x\,/\,(x > 0)\,]\,)$ **do** $\vec{x} \leftarrow \vec{x}$ terminates

$$\varphi:\ -x_1 + 3 \cdot x_3 + 4 > 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q})$: $p > 0$ with

$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of linear loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

- If A is not triangular, transform it into Jordan normal form

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

$\varphi: \; -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \; p > 0$ with

$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

- If A is not triangular, transform it into Jordan normal form

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

$$\varphi : \; -x_1 + 3 \cdot x_3 + 4 > 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}) : \; p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ

$$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \ p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ
- For uniform loops: $\mathrm{red}(p > 0) \iff$ disjunction of *interval conditions* $\mathrm{ic}(p > 0)$

$\varphi : \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}) : \ p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while $(\ \varphi\ )$ **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ
- For uniform loops: $\mathrm{red}(p > 0) \ \Leftrightarrow$ disjunction of *interval conditions* $\mathrm{ic}(p > 0)$

$$x' = 0 \quad \wedge \quad \ldots \quad \wedge \quad x'' = 0 \quad \wedge \quad x \quad > 0$$

$\varphi: \ -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \ p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ
- For uniform loops: $\mathrm{red}(p > 0) \;\Leftrightarrow\;$ disjunction of *interval conditions* $\mathrm{ic}(p > 0)$

$$x' = 0 \quad \wedge \quad \ldots \quad \wedge \quad x'' = 0 \quad \wedge \quad \pm x \quad > 0$$

$\varphi: \; -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \; p > 0$ with

$$p = \frac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\frac{3 \cdot x_4}{2} - \frac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ
- For uniform loops: $\mathrm{red}(p > 0) \;\Leftrightarrow\;$ disjunction of *interval conditions* $\mathrm{ic}(p > 0)$

$$x' = 0 \quad \wedge \quad \ldots \quad \wedge \quad x'' = 0 \quad \wedge \quad \pm x + c > 0$$

$$\varphi: \; -x_1 + 3 \cdot x_3 + 4 > 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\varphi(\vec{q}): \; p > 0$ with

$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

while (φ) **do** $\vec{x} \leftarrow A \cdot \vec{x}$

- If A is not triangular, transform it into Jordan normal form
- A is *uniform* if all eigenvalues $\lambda \in \mathcal{S}_{\geq 0}$ and only one Jordan block for each λ
- For uniform loops: red$(p > 0)$ $\Leftrightarrow$ disjunction of *interval conditions* ic$(p > 0)$

$$\begin{array}{ccccccc} x' = 0 & \wedge & \ldots & \wedge & x'' = 0 & \wedge & \pm x + c > 0 \\ x' = 0 & \wedge & \ldots & \wedge & x'' = 0 & \wedge & x = c \end{array}$$

$\varphi: \; -x_1 + 3 \cdot x_3 + 4 > 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

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$$p = \tfrac{3 \cdot x_5}{8} \cdot n^2 \cdot 2^n + \left(\tfrac{3 \cdot x_4}{2} - \tfrac{3 \cdot x_5}{8}\right) \cdot n \cdot 2^n + 3 \cdot x_3 \cdot 2^n - x_2 \cdot n - x_1 + 4$$

Termination of uniform loops over $\mathcal{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\mathbb{A}}\}$

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Theorem

- Termination of uniform loops is *polynomial* for fixed number k of eigenvalues

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- Termination of uniform loops is *polynomial* for fixed number k of eigenvalues
  $\Rightarrow$ k-termination of uniform loops is in XP

Deciding termination

Theorem [SAS 2020, arXiv 2022]

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Theorem [LPAR 2020, IJCAR 2022]

- Halting problem for twn-loops *decidable*

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Theorem [LPAR 2020, IJCAR 2022]

- Halting problem for twn-loops *decidable*
- Polynomial runtime bounds computable for all twn-loops over $\mathbb{Z}$
  $\Rightarrow$ Combination with incomplete automated complexity analysis in tool KoAT

Theorem [SAS 2020, arXiv 2022]

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Probabilistic programs where termination is decidable

while $(\vec{a} \cdot \vec{x} > \vec{b})$ **do**

p_1: $\vec{x} \leftarrow \vec{x} + \vec{c_1}$

$\vdots$

p_n: $\vec{x} \leftarrow \vec{x} + \vec{c_n}$

p' : $\vec{x} \leftarrow \vec{d}$ // where $\vec{a} \cdot \vec{d} \leq \vec{b}$

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- Giesl, Giesl, Hark (CADE 2019)
  - (Positive) almost-sure termination is decidable

Probabilistic programs where termination and complexity is decidable

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\qquad\qquad \vdots \\
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$$\begin{array}{ll}
\textbf{while } (\vec{a} \cdot \vec{x} > \vec{b}) \textbf{ do} & \\
\quad p_1\colon\ \vec{x} \leftarrow \vec{x} + \vec{c_1} & \\
\qquad\qquad \vdots & \\
\quad p_n\colon\ \vec{x} \leftarrow \vec{x} + \vec{c_n} & \\
\quad p' :\ \vec{x} \leftarrow \vec{d} & // \text{ where } \ \vec{a} \cdot \vec{d} \leq \vec{b}
\end{array}$$

- Giesl, Giesl, Hark (CADE 2019)
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  - asymptotic expected runtime is computable

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