3.2 Herbrand Structures

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3.2 Herbrand-Structures

Now the goal is to check unsatisfiability of a formula in Skolem NF.

$\Rightarrow$ we have to investigate all interpretations $I = (\mathcal{A}, \alpha, \beta)$ and check whether they satisfy the formula.

But: for formulas in Skolem NF, we can restrict ourselves to very special interpretations.

β: not necessary for closed formulas

$\mathcal{A}$: choose $\mathcal{A} := \mathcal{T}(\Sigma)$, i.e., we use the set of all ground terms as domain

α: we fix α_f to be "the function symbol itself".

Now one only has to search for α_p for $p \in \Delta$.

$\Rightarrow$ Search space is much smaller

Def 3.2.1 (Herbrand Structures)

Let (Σ, Δ) be a signature. A Herbrand structure has the form $(\mathcal{T}(\Sigma), \alpha)$ where for all $f \in \Sigma_n$ we have:

$$\alpha_f(t_1, \dots, t_n) = f(t_1, \dots, t_n).$$

If a Herbrand structure is a model of a formula, we call it a Herbrand model.

Ex. 3.2.2. A Herbrand structure for the signature of Ex. 2.12 is: $S = (\mathcal{T}(\Sigma), \alpha)$ with

$\alpha_n = n$ for all $n \in \mathbb{N}$

$\alpha_{monika} = monika, \ldots$

$\alpha_{date}(t_1, t_2, t_3) = date(t_1, t_2, t_3)$ for all $t_1, t_2, t_3 \in \mathcal{T}(\Sigma)$

$\alpha_{female} = \{monika, Karin, \ldots\}$

$\alpha_{male} = \{werner, \ldots\}$

$\alpha_{human} = \mathcal{T}(\Sigma)$

$\alpha_{born} = \{(monika, date(17, 4, 2015)), \ldots\}$

(much nearer to the intuitive semantics)

Looking at H-structures is enough when checking for unsatisfiability of formulas in Skolem NF.

Thm 323 (Satisfiability Check by Herbrand Structures)

Let $\Phi \subseteq \mathcal{F}(\Sigma, \Delta, \mathcal{V})$ be a set of formulas in Skolem NF.

Then Φ is satisfiable iff it has a Herbrand model.

Proof: "$\Leftarrow$": trivial

"$\Rightarrow$": Let $S = (\mathcal{A}, \alpha)$ be a model of Φ.

We now construct a H-structure $S' = (\mathcal{T}(\Sigma), \alpha')$ that is also a model of Φ.

For every $f \in \Sigma_n$, we have $\alpha'_f(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

We define α'_p as follows:

for $p \in \Delta_n$ with $n \geq 1$ we define

$$(t_1, \ldots, t_n) \in \alpha'_p \quad \text{iff} \quad (S(t_1), \ldots, S(t_n)) \in \alpha_p$$

for $p \in \Delta_0$ we define

$$\alpha'_p = \alpha_p$$

Clearly, S' is a H-structure.
It remains to show that for every formula φ in Skolem NF, $S \models \varphi$ implies $S' \models \varphi$.
Since φ is in Skolem NF, it has the form $\forall X_1, \ldots, X_n\ \psi$.
We prove "$S \models \varphi \curvearrowright S' \models \varphi$" by induction on n.

<u>Ind. Base</u>: $n = 0$

Here, φ is quantifier-free.
In this case, we even have $S \models \varphi$ iff $S' \models \varphi$
(easy structural induction on φ).

<u>Ind. Step</u>: $n > 0$

$\forall X_1, \ldots, X_{n-1}\ \psi$ might contain the free var. X_n

Let $S[\![X_n / a]\!]$ denote an interpretation

$$(\underset{\uparrow}{\mathcal{A}}, \underset{\uparrow}{\alpha}, \beta[\![X_n/a]\!]) \text{ for some } \beta.$$

same as for $S = (\mathcal{A}, \alpha)$

Then:

$S \models \forall X_1, \ldots, X_n\ \psi$

$\curvearrowleft\!\!\curvearrowright S[\![X_n/a]\!] \models \forall X_1, \ldots, X_{n-1}\ \psi$ for all $a \in \mathcal{A}$

$\curvearrowright S[\![X_n / S(t)]\!] \models \forall X_1, \ldots, X_{n-1}\ \psi$ for all $t \in \mathcal{T}(\Sigma)$

$\curvearrowleft\!\!\curvearrowright S \models \forall X_1, \ldots, X_{n-1}\ \psi[X_n/t]$ for all $t \in \mathcal{T}(\Sigma)$,

by the subst. lemma 2.2.3.

$\rightsquigarrow S' \models \forall X_1, \dots, X_{n-1}\ \psi[X_n / t]$ for all $t \in \mathcal{T}(\Sigma)$,
by the ind. hypothesis

$\rightsquigarrow S' [\![X_n / \underbrace{S'(t)}_{t,\text{ because } S' \text{ is a } H\text{-structure}}]\!] \models \forall X_1, \dots, X_{n-1}\ \psi$ for all $t \in \mathcal{T}(\Sigma)$

$\rightsquigarrow S' \models \forall X_1, \dots, X_n\ \psi$ $\boxed{\lightning}$

Ex. 324 Thm 323 only holds for formulas in Skolem NF.

Consider $\Phi = \{ p(a), \exists X \neg p(X) \}$.

Φ is satisfiable, but it has no Herbrand model over the signature (Σ, Δ) where $\Sigma = \Sigma_0 = \{a\}$ and $\Delta = \Delta_1 = \{p\}$.

The following structure S is a model of Φ:

$S = (\{0,1\}, \alpha)$ where $\alpha_a = 0$
$\alpha_p = \{0\}$

$S \models p(a) \qquad S \models \exists X\ \neg p(X)$

but there is an element in the domain of S that does not correspond to any ground term:

$S(t) \neq 1$ for all $t \in \mathcal{T}(\Sigma)$

Since $\mathcal{T}(\Sigma) = \{a\}$, any H-structure S' has the domain $\{a\}$ and therefore $S' \models p(a)$ implies $S' \not\models \exists X\ \neg p(X)$.

For formulas in Skolem NF:

$$\forall X_1, \dots, X_n \quad \psi$$

One only has to instantiate $X_1, \dots, X_n$ by all possible ground terms and check whether all of the resulting formulas are satisfiable.

<u>Def 3.25</u> (Herbrand-expansion of a formula)
Let $\varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$ be a formula in Skolem NF, i.e., $\varphi = \forall X_1, \dots, X_n \ \psi$ where ψ is quantifier-free.
The following set of formulas $E(\varphi)$ is called the <u>Herbrand-expansion</u> of φ:

$$E(\varphi) = \{ \psi[X_1/t_1, \dots, X_n/t_n] \mid t_1, \dots, t_n \in \mathcal{T}(\Sigma) \}$$

(i.e., it is the set of all <u>ground instances</u> of ψ).

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terms (syntactic)

$\varphi[X_1/t_1, \dots, X_n/t_n]$ is φ with a <u>substitution</u> mapping X_i to t_i

$I[\![X_1/a_1, \dots, X_n/a_n]\!]$ is an interpretation with a <u>variable assignment</u> assigning a_i to X_i

elements of the domain (semantic)

<u>Ex. 3.26</u> To prove the query ?– motherOf(X, susanne).

one has to prove unsatisfiability (cf. Ex. 3.1.4.)

$\varphi = \forall X\,(\text{motherOf}(\text{renate}, \text{susanne}) \wedge \neg\,\text{motherOf}(X, \text{susanne}))$.

$E(\varphi) = \{\ \text{mO}(\text{ren}, \text{sus}) \wedge \neg\,\text{mO}(\text{karin}, \text{susanne}),$

$\underline{\text{mO}(\text{ren}, \text{sus}) \wedge \neg\,\text{mO}(\text{ren}, \text{sus})},$

$\text{mO}(\text{ren}, \text{sus}) \wedge \neg\,\text{mO}(\text{date}(17, 4, 2015), \text{sus}),$

$\vdots$

$\}$

We will see that

φ is satisfiable iff $E(\varphi)$ is satisfiable

Since the red subformula is unsat.

$\Rightarrow E(\varphi)$ is unsat

$\Rightarrow \varphi$ is unsat

$\Rightarrow$ query is true.

Thm 3.2.7 (Satisfiability of Herbrand-expansion)

Let φ be a formula in Skolem NF.

Then φ is satisfiable iff $E(\varphi)$ is satisfiable.

Proof: φ has the form $\forall X_1, \dots, X_n\ \psi$ where ψ is quantifier-free.

φ is satisfiable

$\forall X_1, \dots, X_n\ \psi$

iff there is a Herbrand-structure S with

$S \models \forall X_1, \dots, X_n\ \psi$ (Thm 3.2.3)

iff there is a H-str. S with

$S[\![X_1/t_1, \dots, X_n/t_n]\!] \models \psi$ for all $t_1, \dots, t_n \in \mathcal{T}(\Sigma)$

iff there is a H-str. S with

$S \models \psi[X_1/t_1, \dots, X_n/t_n]$ for all $t_1, \dots, t_n \in \mathcal{T}(\Sigma)$

(by the subst. lemma 2.2.3)

iff there is a H-str. S with

$S \models E(\varphi)$

iff $E(\varphi)$ is satisfiable

$\boxtimes$

For a formula φ in Skolem NF:

To check whether φ is unsatisfiable, we can construct $E(\varphi)$ and check whether some finite subset of $E(\varphi)$ is unsatisfiable.

(Compactness Theorem: if an infinite set of formulas is unsatisfiable, then there is already a finite subset that is unsatisfiable).

$\Rightarrow$ Algorithm of Gilmore

First semi-decision procedure for entailment/unsatisfiability.

Formulas without variables correspond to propositional logic:

- every occurring atomic sub-formula corresponds to a propositional variable (i.e., it can be either TRUE or FALSE)

- by trying out all truth assignments for these prop. variables, one can decide satisfiability of formulas without variables

Ex 328 We wanted to check satisfiability of

$$E(\varphi) = \{ \underbrace{mO(ren, sus)}_{V_{mo(r,s)}} \wedge \neg \underbrace{mO(Karin, sus)}_{V_{mo(K,s)}},$$
$$\underbrace{mO(ren, sus)}_{V_{mo(r,s)}} \wedge \neg \underbrace{mO(ren, sus)}_{V_{mo(r,s)}}, \dots \}$$

One can replace all atomic sub-formulas by propositional variables:

$$\{ V_{mo(r,s)} \wedge \neg V_{mo(K,s)},$$
$$V_{mo(r,s)} \wedge \neg V_{mo(r,s)}, \dots \}$$

Then construct finite subsets step by step and check satisfiability by mapping $V_{\dots}$ to {TRUE, FALSE}