2.2 Semantics of Predicate Logic

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We now determine which objects are described by terms and when a formula is true or false.

Interpretation:

- determines a set of objects that we speak about (<u>carrier</u> $\mathcal{A}$)
- maps every function symbol f to a function α_f
  — " — pred. symbol p to a relation α_p
- assigns objects to variables (variable assignment β).

<u>Def</u> 2.2.1 (Interpretation, Structure) — Slide 11

For a signature (Σ, Δ), an interpretation has the form $I = (\mathcal{A}, \alpha, \beta)$.

$\mathcal{A}$ is an arbitrary set with $\mathcal{A} \neq \emptyset$ (<u>carrier</u>).

α maps every $f \in \Sigma_n$ to a function $\alpha_f : \underbrace{\mathcal{A} \times \ldots \times \mathcal{A}}_{n \text{ times}} \to \mathcal{A}$

α maps every $p \in \Delta_n$ to a $\alpha_p \subseteq \underbrace{\mathcal{A} \times \ldots \times \mathcal{A}}_{n \text{ times}}$ if $n \geq 1$

For $p \in \Delta_0$, $\alpha_p \in \{TRUE, FALSE\}$

We say that α_f resp. α_p is the <u>meaning</u> of f resp. p.

$\beta : \mathcal{V} \to \mathcal{A}$ is called a <u>variable assignment</u>.

Every interpretation I gives a meaning to every term, i.e. $I : \mathcal{T}(\Sigma, \mathcal{V}) \to \mathcal{A}$ as follows:

$$\mathcal{I}(X) = \beta(X)$$

$$\mathcal{I}(f(t_1, \dots, t_n)) = \alpha_f(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n))$$

Ex. 2.2.2 Example Interpretation $\mathcal{I} = (\mathcal{A}, \alpha, \beta)$ with

$\mathcal{A} = \mathbb{N}$

$\alpha_n = n$ for all $n \in \mathbb{N}$

$\alpha_{monika} = 0$

$\alpha_{Karin} = 1$

$\alpha_{renate} = 2$

$\vdots$

$\alpha_{date}(n_1, n_2, n_3) = n_1 + n_2 + n_3$

$\alpha_{female} = \{ n \mid n \text{ is even} \}$

$\alpha_{male} = \{ n \mid n \text{ is odd} \}$

$\alpha_{human} = \mathbb{N}$

$\alpha_{married} = \{ (n, m) \mid n > m \}$

$\vdots$

$\beta(X) = 0$

$\beta(Y) = 1$

$\beta(Z) = 2$

$\vdots$

Then $\mathcal{I}(date(1, X, karin)) = \alpha_{date}(\alpha_1, \beta(X), \alpha_{karin})$

$= 1 + 0 + 1 = 2$

Def 2.2.1 (cont.)

For $X \in \mathcal{V}$ and $a \in \mathcal{A}$, let $\beta[\![X/a]\!]$ be the variable assignment with $\beta[\![X/a]\!](X) = a$ and $\beta[\![X/a]\!](Y) = \beta(Y)$ for all $Y \in \mathcal{V}$ with $Y \neq X$.

For $\mathcal{I} = (\mathcal{A}, \alpha, \beta)$, let $\mathcal{I}[\![X/a]\!] = (\mathcal{A}, \alpha, \beta[\![X/a]\!])$.

An interpretation $\mathcal{I} = (\mathcal{A}, \alpha, \beta)$ satisfies a formula $\varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$, denoted "$\mathcal{I} \models \varphi$", iff

if and

- $\varphi = p(t_1, \dots, t_n)$ and $(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n)) \in \alpha_p$

- $\varphi = p(t_1,\dots,t_n)$ and $(I(t_1),\dots,I(t_n)) \in \alpha_p$ if $p \in \Delta_n$ and $n \geq 1$

↑ if and only if

- $\varphi = p$ and $\alpha_p = TRUE$ if $p \in \Delta_0$
- $\varphi = \neg\varphi_1$ and $I \not\models \varphi_1$
- $\varphi = \varphi_1 \wedge \varphi_2$ and $I \models \varphi_1$ and $I \models \varphi_2$
- $\varphi = \varphi_1 \vee \varphi_2$ and ($I \models \varphi_1$ or $I \models \varphi_2$)
- $\varphi = \varphi_1 \rightarrow \varphi_2$ and if $I \models \varphi_1$, then also $I \models \varphi_2$
- $\varphi = \varphi_1 \leftrightarrow \varphi_2$ and ($I \models \varphi_1$ iff $I \models \varphi_2$)
- $\varphi = \forall X\ \varphi_1$ and $I[\![X/a]\!] \models \varphi_1$ for all $a \in \mathcal{A}$
- $\varphi = \exists X\ \varphi_1$ and there exists $a \in \mathcal{A}$ such that $I[\![X/a]\!] \models \varphi_1$

Ex 2.2.2 (cont.)

$I \models married(date(1, X, karin), karin)$, because

$$\Big(\underbrace{I(date(1,X,karin))}_{2},\ \underbrace{I(karin)}_{\alpha_{karin}=1} \Big) \in \underbrace{\alpha_{married}}_{\{(n,m)\,|\,n>m\}}$$

$I \models \forall X\ female(date(X,X,monika))$, because

$$\underbrace{I[\![X/a]\!](date(X,X,monika))}_{\underbrace{\alpha_{date}(a,\ a,\ \alpha_{monika})}_{}} \in \underbrace{\alpha_{female}}_{\{n\,|\,n \text{ is even}\}}$$

$a + a + 0$

holds for all $a \in \underset{\mathbb{N}}{\mathcal{A}}$

Def 2.2.1 (cont.)

An interpretation I is a model of φ iff $I \models \varphi$.

I is a model of a set of formulas Φ iff
$I \models \varphi$ for all $\varphi \in \Phi$.

Two formulas φ_1, φ_2 are equivalent iff we have
$(I \models \varphi_1$ iff $I \models \varphi_2)$ for all interpretations I.

A formula (resp. a set of formulas) is satisfiable iff it has a model.

A formula is valid if every interpretation is a model of the formula.

tautology (allgemeingültig)

- married(gerd, susanne) is satisfiable, not valid
- married $(2,1) \vee \neg$ married $(2,1)$ is valid
- $\varphi \wedge \neg\varphi$ is unsatisfiable for any formula φ

An interpretation without variable assignment is called a structure $S = (\mathcal{A}, \alpha)$.

If we only regard closed formulas, then satisfiability etc. can be defined for structures:

$S \models \varphi$ iff $I \models \varphi$ for some interpretation

$$I = (\mathcal{A}, \alpha, \beta)$$

Similarly, we can define $S(t)$ for ground terms t.

2 similar concepts:

Substitution $\sigma : \mathcal{V} \to \mathcal{T}(\Sigma, \mathcal{V})$ (syntactic)

Variable assignment $\beta : \mathcal{V} \to \mathcal{A}$ (semantic)

Lemma 2.2.3 (Substitution Lemma)

Let $I = (\mathcal{A}, \alpha, \beta)$ be an interpretation for a signature (Σ, Δ), let $\sigma = \{X_1/t_1, \ldots, X_n/t_n\}$. Then we have

(a) $I(\sigma(t)) = I[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!](t)$ for all $t \in \mathcal{T}(\Sigma, \mathcal{V})$

(b) $I \models \sigma(\varphi)$ iff $I[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!] \models \varphi$

for all $\varphi \in \mathcal{F}(\Sigma, \mathcal{V})$.

Ex 2.2.4 Let I be the interpr. from Ex. 2.2.2.

let $\sigma = \{X / date(1, X, karin)\}$, let $t = date(X, Y, Z)$.

$$I(\sigma(t)) = I(date(date(1, X, karin), Y, Z))$$
$$= \alpha_{date}(\alpha_{date}(\alpha_1, \beta(X), \alpha_{karin}), \beta(Y), \beta(Z))$$
$$= 1 + 0 + 1 + 1 + 2 = 5$$

$$I[\![X / \underbrace{I(date(1, X, karin))}_{2}]\!](date(X, Y, Z)) =$$

$$\alpha_{date}(\beta[\![X/2]\!](X), \beta[\![X/2]\!](Y), \beta[\![X/2]\!](Z)) =$$
$$2 + 1 + 2 = 5$$

Proof of Lemma 2.2.3 (Substitution Lemma)

$$\mathcal{I}(\sigma(t)) = \mathcal{I}[\![X_1/\sigma(t_1), \dots, X_n/\sigma(t_n)]\!](t)$$

for all terms t.

Proof by structural induction on t (Induction corresponds to the definition of the data structure for terms).

Ind. Base: prove the statement for the smallest possible terms (i.e. $t \in \mathcal{V} \cup \Sigma_0$)

Ind. Step: prove the statement for terms of the form $f(s_1, \dots, s_k)$

Ind. Hypothesis: statement already holds for smaller terms $s_1, \dots, s_k$.

Ind. Base $\quad t$ is a variable (The case where $t \in \Sigma_0$ is handled together with the ind. step.)

$\sigma = \{X_1/t_1, \dots, X_n/t_n\}$.

Case 1: $t = X_i$

$$\mathcal{I}(\sigma(X_i)) = \mathcal{I}(t_i)$$

$$\mathcal{I}[\![X_1/\mathcal{I}(t_1), \dots, X_n/\mathcal{I}(t_n)]\!](X_i) = \mathcal{I}(t_i)$$

Case 2: t is a variable $Y \notin \{X_1, \ldots, X_n\}$

$$I(\sigma(Y)) = I(Y) = \beta(Y)$$

$$I[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!](Y) = \beta[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!](Y) = \beta(Y)$$

Ind. Step: $t = f(s_1, \ldots, s_k)$ with $k \geq 0$

$$I(\sigma(t)) = I(\sigma(f(s_1, \ldots, s_k))) = I(f(\sigma(s_1), \ldots, \sigma(s_k))) = \alpha_f(\underline{I(\sigma(s_1))}, \ldots, \underline{I(\sigma(s_k))})$$

$$I[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!](t) = I[\![\ldots]\!](f(s_1, \ldots, s_k)) = \alpha_f(\underline{I[\![X_1/I(t_1), \ldots, X_n/I(t_n)]\!](s_1)}, \ldots, \underline{I[\![\ldots]\!](s_k)})$$

equal due to the ind. hypothesis

equal due to the ind. hypothesis

The proof of (b) is similar, by structural induction on formulas. $\boxtimes$

Def 2.2.5 (Entailment)

A set of formulas Φ <u>entails</u> a formula φ (denoted

$\Phi \models \varphi$) iff for all interpretations I with $I \models \Phi$ we also have $I \models \varphi$.
(If Φ and φ have no free variables, then $\Phi \models \varphi$ means that $S \models \Phi$ implies $S \models \varphi$ for all structures S.)

So "$\models$" has two meanings:

$I \models \varphi$ means: the interpr. I satisfies φ

$\Phi \models \varphi$ means: Φ entails φ

Instead of $\emptyset \models \varphi$, we also write $\models \varphi$.

empty set of formulas

means: φ is valid

Ex. 2.2.6

Let Φ be the set of formulas from Ex. 2.1.7 that corresponds to the logic prog. from Chapter 1.
Asking the query

?- male(gerd).

means that one has to prove

$\Phi \models$ male(gerd).

This clearly holds, because male(gerd) $\in \Phi$.

Asking ?- human(gerd).

Asking ?- human(gerd).

means that one has to prove

$$\Phi \models human(gerd).$$

This holds, as Φ contains $\forall X \; human(X)$:

$$I \models \forall X \; human(X)$$

$\curvearrowright$

$$I[\![X/a]\!] \models human(X) \text{ for all } a \in \mathcal{A}$$

$\curvearrowright$

$$I[\![X/\alpha_{gerd}]\!] \models human(X)$$

$\curvearrowright$ by the Subst. Lemma

$$I \models human(gerd)$$

Asking the query ?- motherOf(X, susanne)

means that one has to prove

$$\Phi \models \exists X \; motherOf(X, susanne)$$

How does Prolog do this automatically?