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Notes:

- Please solve these exercises in **groups of three or four**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Friday, June 23rd, 2017, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (this box is emptied **15 minutes before** the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all students on your solution. Also please staple the individual sheets!

Exercise 1 (Semantics): (3 + 3 + 3 + 1 = 10 points)

Consider the following logic program $\mathcal{P}$ over the signature (Σ, Δ) with $\mathtt{0}, \mathtt{s} \in \Sigma$ and $\mathtt{minus}, \mathtt{pred} \in \Delta$:

```
minus(X, 0, X).
minus(X, Y, Z) :- minus(A, B, Z), pred(X, A), pred(Y, B).
pred(s(s(X)), s(Y)) :- pred(s(X), Y).
pred(s(X), X).
```

Also consider the following query:

```
?- minus(s(s(s(0))), Y, s(0)).
```

Hints:

- In each of the computation steps that you give in this exercise, you can abbreviate those parts of the substitution that did not change by „...“.

a) Show a successful computation for the query above (i.e., a computation of the form $(G, \emptyset) \vdash_{\mathcal{P}}^{+} (\Box, \sigma)$ where $G = \{\neg \mathtt{minus}(\mathtt{s}^3(\mathtt{0}), Y, \mathtt{s}(\mathtt{0}))\}$). Also give the answer substitution.

b) Show a finite unsuccessful computation for the query above (i.e., a computation of the form $(G, \emptyset) \vdash_{\mathcal{P}}^{+} (G_1, \sigma_1)$ where $G = \{\neg \mathtt{minus}(\mathtt{s}^3(\mathtt{0}), Y, \mathtt{s}(\mathtt{0}))\}$, $G_1 \neq \Box$ and there are no G_2 and σ_2 such that $(G_1, \sigma_1) \vdash_{\mathcal{P}} (G_2, \sigma_2)$).

c) Indicate an infinite computation for the query above by giving the first few steps. Give enough steps so that it is obvious how the infinite computation looks like.

d) What is $D[\![\mathcal{P}, G]\!]$ in this example?

Example: The query
`?- pred(s(s(0)), Y).`
has two successful derivations. Here we use variable renamings to replace the variables X and Y in the program clauses by X_1 and Y_1 resp. X_2:

$$(\{\underline{\neg \mathtt{pred}(\mathtt{s}(\mathtt{s}(\mathtt{0})), Y)}\}, \emptyset) \vdash_{\mathcal{P}} (\{\underline{\neg \mathtt{pred}(\mathtt{s}(\mathtt{0}), Y_1)}\}, \{X_1/\mathtt{0}, Y/\mathtt{s}(Y_1)\}) \quad (1)$$
$$\vdash_{\mathcal{P}} (\Box, \{X_1/\mathtt{0}, Y/\mathtt{s}(\mathtt{0}), X_2/\mathtt{0}, Y_1/\mathtt{0}\})$$

$$(\{\underline{\neg \mathtt{pred}(\mathtt{s}(\mathtt{s}(\mathtt{0})), Y)}\}, \emptyset) \vdash_{\mathcal{P}} (\Box, \{X_1/\mathtt{s}(\mathtt{0}), Y/\mathtt{s}(\mathtt{0})\}) \quad (2)$$

The answer substitution for both derivations is $\{Y/\mathtt{s}(\mathtt{0})\}$.

Exercise 2 (Fixpoints): (3 + 3 + 3 = 9 points)

Consider the function $f\colon Pot(\mathbb{N}) \to Pot(\mathbb{N})$.

$$f(M) = \begin{cases} \{0\}, & \text{if } M = \emptyset \\ \{(\sum\limits_{y \in X} y) - \min X \mid \emptyset \neq X \subseteq M\}, & \text{if } M \neq \emptyset \text{ is finite} \\ \mathbb{N}, & \text{otherwise} \end{cases}$$

So for example, $f(\{2,5\}) = \{(\sum\limits_{y \in \{2\}} y) - \min\{2\}, (\sum\limits_{y \in \{5\}} y) - \min\{5\}, (\sum\limits_{y \in \{2,5\}} y) - \min\{2,5\}\} = \{0,5\}$.

a) Prove that f is monotonic.

b) Prove or disprove that f is continuous.

c) Please give all fixpoints of f and mark the least fixpoint.

Exercise 3 (Fixpoint Semantics): (3 + 1 + 1 + 1 = 6 points)

Reconsider the logic program $\mathcal{P}$ over the signature (Σ, Δ) with $\texttt{0}, \texttt{s} \in \Sigma$ and $\texttt{minus}, \texttt{pred} \in \Delta$ from Exercise 1, where we remove the recursive rule of the predicate `pred`:

```
minus(X, 0, X).
minus(X, Y, Z) :- minus(A, B, Z), pred(X, A), pred(Y, B).
pred(s(X), X).
```

a) For each $i \in \mathbb{N}$ explicitly give $\underline{\text{trans}}_{\mathcal{P}}^{i}(\emptyset)$.

b) Compute the set $\text{lfp}(\underline{\text{trans}}_{\mathcal{P}})$.

c) Give $F[\![\mathcal{P}, \{\neg\texttt{minus(s(s(0)), Y, Z)}\}]\!]$.

d) Give $F[\![\mathcal{P}, \{\neg\texttt{minus(X, s(Y), Y)}\}]\!]$.

Exercise 4 (Proofs): (2 + 2 = 4 points)

a) Please show that every continuous function $f\colon \mathcal{P}ot(M) \to \mathcal{P}ot(M)$ is monotonic.

b) Please show that for every *finite* chain

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$$

and every monotonic function $f\colon \mathcal{P}ot(M) \to \mathcal{P}ot(M)$ we have:

$$f(\bigcup_{i=1}^{n} M_i) = \bigcup_{i=1}^{n} f(M_i)$$