Exercise 1 (Herbrand model): (2 + 2 = 4 points)

Consider again the formula

$$\begin{aligned}\varphi = &\forall X\ \mathsf{p}(X,X)\\ &\wedge (\forall X,Y\ \mathsf{p}(X,Y) \to \mathsf{p}(X,\mathsf{f}(Y)))\\ &\wedge \neg\forall X\ \mathsf{p}(\mathsf{a},X)\end{aligned}$$

from Exercise Sheet 2 over the signature (Σ,Δ) with $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{\mathsf{a}\}, \Sigma_1 = \{\mathsf{f}\}$, $\Delta = \Delta_2 = \{\mathsf{p}\}$, and its Skolem normal form

$$\psi = \ \forall X_1, X_2, Y \quad \mathsf{p}(X_1,X_1) \wedge (\neg\mathsf{p}(X_2,Y) \vee \mathsf{p}(X_2,\mathsf{f}(Y))) \wedge \neg\mathsf{p}(\mathsf{a},\mathsf{g}(X_1,X_2,Y))$$

Hints:

- You can use $\mathsf{f}^i(\mathsf{a})$ as an abbreviation for $\underbrace{\mathsf{f}(\ldots(\mathsf{f}}_{i \text{ times}}(\mathsf{a}))\ldots)$.

a) Give a Herbrand model for φ or show why no such model exists.

b) Give a Herbrand model for ψ or show why no such model exists.

Solution:

a) Assume there is a Herbrand model $S = (\mathcal{A},\alpha)$ with $S \models \varphi$. Then $\mathcal{A} = \{\mathsf{a}\} \cup \{\mathsf{f}^i(\mathsf{a}) \mid i \in \mathbb{N}, i \geq 1\}$. Due to the first subformula, we know that $\mathsf{p}(\mathsf{a},\mathsf{a})$ holds. Because of $S \models \mathsf{p}(\mathsf{a},\mathsf{a})$ and $S \models \forall X,Y\ \mathsf{p}(X,Y) \to \mathsf{p}(X,\mathsf{f}(Y))$, we know $S \models \mathsf{p}(\mathsf{a},\mathsf{f}(\mathsf{a}))$. By induction, we have $\{(\mathsf{a},\mathsf{a})\} \cup \{(\mathsf{a},\mathsf{f}^i(\mathsf{a})) \mid i \in \mathbb{N}, i \geq 1\} \subseteq \alpha_{\mathsf{p}}$. This contradicts the last subformula, since one can only assign terms from $\mathcal{A} = \{\mathsf{a}\} \cup \{\mathsf{f}^i(\mathsf{a}) \mid i \in \mathbb{N}, i \geq 1\}$ to X.

b) $S' = (\mathcal{A}',\alpha')$ with $\mathcal{A}' = \mathcal{T}(\Sigma \cup \{\mathsf{g}\})$ and

- $\alpha'_{\mathsf{a}} = \mathsf{a}$,
- $\alpha'_{\mathsf{f}}(t) = \mathsf{f}(t)$ for all $t \in \mathcal{T}(\Sigma \cup \{\mathsf{g}\})$,
- $\alpha'_{\mathsf{g}}(t_1,t_2,t_3) = \mathsf{g}(t_1,t_2,t_3)$ for all $t_1,t_2,t_3 \in \mathcal{T}(\Sigma \cup \{\mathsf{g}\})$,
- $\alpha'_{\mathsf{p}} = \{(t_1,t_2) \mid t_1,t_2 \in \mathcal{T}(\Sigma \cup \{\mathsf{g}\}),\ \text{if } t_1 = \mathsf{a} \text{ then } \operatorname{root}(t_2) \neq \mathsf{g}\}$.

Exercise 2 (Gilmore's algorithm): (4 points)

Consider the following logic program

```
equals(0,0).
equals(s(X),s(Y)) :- equals(X,Y).
```

and the query

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? - equals(s(s(0)), s(s(0))).
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Using Gilmore's algorithm show that the formulas φ_1 and φ_2 corresponding to the logic program entail the formula φ corresponding to the query (i.e., $\{\varphi_1,\varphi_2\} \models \varphi$).

Solution:

The formulas corresponding to the program are $\varphi_1 := \mathtt{equals}(0,0)$ and $\varphi_2 = \forall X,Y\,\mathtt{equals}(X,Y) \to$

$\mathtt{equals}(\mathsf{s}(X), \mathsf{s}(Y))$. The formula corresponding to the query is $\varphi := \mathtt{equals}(\mathsf{s}(\mathsf{s}(0)), \mathsf{s}(\mathsf{s}(0)))$. We need to show that $\varphi_1 \wedge \varphi_2 \wedge \neg\varphi$ is unsatisfiable. The corresponding formula in Skolem normal form is $\forall X, Y \psi$ with

$$\psi = \mathtt{equals}(0,0) \wedge (\mathtt{equals}(X,Y) \to \mathtt{equals}(\mathsf{s}(X),\mathsf{s}(Y))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))$$

With the following enumeration of the Herbrand expansion (where $\mathsf{s}^i(0) = \underbrace{\mathsf{s}(\ldots(\mathsf{s}}_{i \text{ times}}(0))\ldots))$

$$\begin{aligned}
\psi_1 = \psi[X/0, Y/0] &= \mathtt{equals}(0,0) \wedge (\mathtt{equals}(0,0) \to \mathtt{equals}(\mathsf{s}(0),\mathsf{s}(0))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))\\
\psi_2 = \psi[X/0, Y/\mathsf{s}(0)] &= \mathtt{equals}(0,0) \wedge (\mathtt{equals}(0,\mathsf{s}(0)) \to \mathtt{equals}(\mathsf{s}(0),\mathsf{s}^2(0))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))\\
\psi_3 = \psi[X/\mathsf{s}(0), Y/0] &= \mathtt{equals}(0,0) \wedge (\mathtt{equals}(\mathsf{s}(0),0) \to \mathtt{equals}(\mathsf{s}^2(0),\mathsf{s}(0))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))\\
\psi_4 = \psi[X/0, Y/\mathsf{s}^2(0)] &= \mathtt{equals}(0,0) \wedge (\mathtt{equals}(0,\mathsf{s}^2(0)) \to \mathtt{equals}(\mathsf{s}(0),\mathsf{s}^3(0))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))\\
\psi_5 = \psi[X/\mathsf{s}(0), Y/\mathsf{s}(0)] &= \mathtt{equals}(0,0) \wedge (\mathtt{equals}(\mathsf{s}(0),\mathsf{s}(0)) \to \mathtt{equals}(\mathsf{s}^2(0),\mathsf{s}^2(0))) \wedge \neg\mathtt{equals}(\mathsf{s}(\mathsf{s}(0)),\mathsf{s}(\mathsf{s}(0)))\\
\ldots
\end{aligned}$$

and by replacing all atomic formulas with propositional variables (e.g. A_{01} for $\mathtt{equals}(0,\mathsf{s}(0))$) we get:

$$\begin{aligned}
\psi'_1 &= A_{00} \wedge (A_{00} \to A_{11}) \wedge \neg A_{22}\\
\psi'_2 &= A_{00} \wedge (A_{01} \to A_{12}) \wedge \neg A_{22}\\
\psi'_3 &= A_{00} \wedge (A_{10} \to A_{21}) \wedge \neg A_{22}\\
\psi'_4 &= A_{00} \wedge (A_{02} \to A_{13}) \wedge \neg A_{22}\\
\psi'_5 &= A_{00} \wedge (A_{11} \to A_{22}) \wedge \neg A_{22}\\
&\ldots
\end{aligned}$$

When considering $\psi'_1 \wedge \ldots \wedge \psi'_5$, the subformula $A_{00} \wedge (A_{00} \to A_{11}) \wedge (A_{11} \to A_{22}) \wedge \neg A_{22}$ is unsatisfiable.

Exercise 3 (Conjunctive Normal Form): (2 points)

Consider the following formula φ with $\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3 \in \Delta_0$:

$$\varphi = (\neg(\mathsf{p}_1 \to \mathsf{p}_2) \vee (\neg\mathsf{p}_2 \to \mathsf{p}_3)) \vee \mathsf{p}_3$$

Use the algorithm presented in the proof of Theorem 3.3.2 to convert φ to an equivalent formula in *conjunctive normal form (CNF)*.

Solution:

$$\begin{aligned}
\varphi &= (\neg(\mathsf{p}_1 \to \mathsf{p}_2) \vee (\neg\mathsf{p}_2 \to \mathsf{p}_3)) \vee \mathsf{p}_3\\
&\equiv (\neg(\neg\mathsf{p}_1 \vee \mathsf{p}_2) \vee (\mathsf{p}_2 \vee \mathsf{p}_3)) \vee \mathsf{p}_3\\
&\equiv ((\mathsf{p}_1 \wedge \neg\mathsf{p}_2) \vee (\mathsf{p}_2 \vee \mathsf{p}_3)) \vee \mathsf{p}_3\\
&\equiv ((\mathsf{p}_1 \vee \mathsf{p}_2 \vee \mathsf{p}_3) \wedge (\neg\mathsf{p}_2 \vee \mathsf{p}_2 \vee \mathsf{p}_3)) \vee \mathsf{p}_3\\
&\equiv (\mathsf{p}_1 \vee \mathsf{p}_2 \vee \mathsf{p}_3 \vee \mathsf{p}_3) \wedge (\neg\mathsf{p}_2 \vee \mathsf{p}_2 \vee \mathsf{p}_3 \vee \mathsf{p}_3)
\end{aligned}$$

Exercise 4 (Resolution for Propositional Logic): (3 points)

Consider the following clause set $\mathcal{K}$ with $\mathsf{p}_1, \ldots, \mathsf{p}_4 \in \Delta_0$:

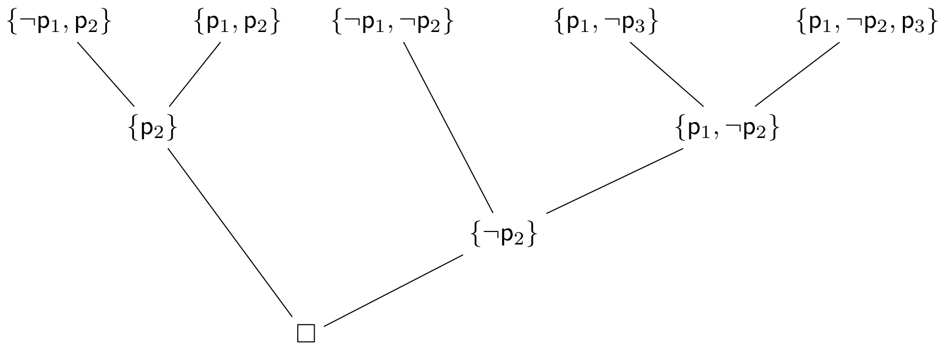
$$\mathcal{K} = \{\{\neg\mathsf{p}_1, \mathsf{p}_2\}, \{\mathsf{p}_1, \mathsf{p}_2\}, \{\neg\mathsf{p}_1, \neg\mathsf{p}_2\}, \{\mathsf{p}_1, \neg\mathsf{p}_3\}, \{\mathsf{p}_1, \neg\mathsf{p}_2, \mathsf{p}_3\}\}$$

Please show that $\mathcal{K}$ is unsatisfiable by using resolution for propositional logic (cf. Definition 3.3.4 and Example 3.3.5).

Hints:

- It suffices to perform four resolution steps.

Solution:



Since we have $\Box \in Res^*(\mathcal{K})$, we can conclude that $\mathcal{K}$ is unsatisfiable.

Exercise 5 (Unification): (2 + 2 + 2 = 6 points)

Consider the signature (Σ, Δ) with $\Sigma_0 = \{a, b\}, \Sigma_1 = \{f\}, \Sigma_2 = \{g\}$ and $\Delta_3 = \{p\}$. Use the algorithm from the lecture to decide whether the following clauses are unifiable. To document your application of the algorithm on some clause K, please write down the current substituted clause $\sigma(K)$ whenever the algorithm checks whether $|\sigma(K)| = 1$ and underline the position of the next symbols where the literals are not equal. Additionally, write down the resulting most general unifier (mgu) or the kind of failure (clash or occur) the algorithm returns.

a) $\{p(X, f(Y), g(f(a), b)), p(f(Z), Z, g(X, b))\}$

b) $\{p(X, g(a, f(X)), Y), p(Z, g(a, Y), Z)\}$

c) $\{p(g(X, Y), f(Z), Z), p(W, Y, f(a))\}$

Hint: To illustrate this exercise, we give a short example for the clause $\{p(X, Y, Z), p(Z, a, b)\}$:

1. $\{p(\underline{X}, f(Y), Z), p(\underline{Z}, f(a), b)\}$
2. $\{p(Z, f(\underline{Y}), Z), p(Z, f(\underline{a}), b)\}$
3. $\{p(Z, f(a), \underline{Z}), p(Z, f(a), \underline{b})\}$
4. $\{p(b, f(a), b)\}$
5. mgu: $\{X/b, Y/a, Z/b\}$

Solution:

a) 1. $\{p(\underline{X}, f(Y), g(f(a), b)), p(\underline{f(Z)}, Z, g(X, b))\}$
2. $\{p(f(Z), \underline{f(Y)}, g(f(a), b)), p(f(Z), \underline{Z}, g(f(Z), b))\}$
3. $\{p(f(f(Y)), f(Y), g(f(\underline{a}), b)), p(f(f(Y)), f(Y), g(f(\underline{f(Y)}), b))\}$
4. clash failure $(a = f(\ldots))$

b) 1. $\{p(\underline{X}, g(a, f(X)), Y), p(\underline{Z}, g(a, Y), Z)\}$
2. $\{p(Z, g(a, \underline{f(Z)}), Y), p(Z, g(a, \underline{Y}), Z)\}$
3. $\{p(Z, g(a, f(Z)), \underline{f(Z)}), p(Z, g(a, f(Z)), \underline{Z})\}$

4. occur failure ($Z = \mathsf{f}(Z)$)

c)
1. $\{\mathsf{p}(\underline{\mathsf{g}(X, Y)}, \mathsf{f}(Z), Z), \mathsf{p}(\underline{W}, Y, \mathsf{f}(\mathsf{a}))\}$
2. $\{\mathsf{p}(\mathsf{g}(X, Y), \underline{\mathsf{f}(Z)}, Z), \mathsf{p}(\mathsf{g}(X, Y), \underline{Y}, \mathsf{f}(\mathsf{a}))\}$
3. $\{\mathsf{p}(\mathsf{g}(X, \mathsf{f}(Z)), \mathsf{f}(Z), \underline{Z}), \mathsf{p}(\mathsf{g}(X, \mathsf{f}(Z)), \mathsf{f}(Z), \underline{\mathsf{f}(\mathsf{a})})\}$
4. $\{\mathsf{p}(\mathsf{g}(X, \mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{a})), \mathsf{f}(\mathsf{a}))\}$
5. mgu: $\{W/\mathsf{g}(X, \mathsf{f}(\mathsf{f}(\mathsf{a}))), Y/\mathsf{f}(\mathsf{f}(\mathsf{a})), Z/\mathsf{f}(\mathsf{a})\}$