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Exercise 1 (Resolution): (2 points)

Consider again the following logic program from Exercise Sheet 3

```
equals(0,0).
equals(s(X),s(Y)) :- equals(X,Y).
```

and the query

```
? - equals(s(s(0)),s(s(0))).
```

Show that the formulas φ_1 and φ_2 corresponding to the logic program entail the formula φ corresponding to the query (i.e., $\{\varphi_1, \varphi_2\} \models \varphi$) using the resolution algorithm in predicate logic.

Solution:

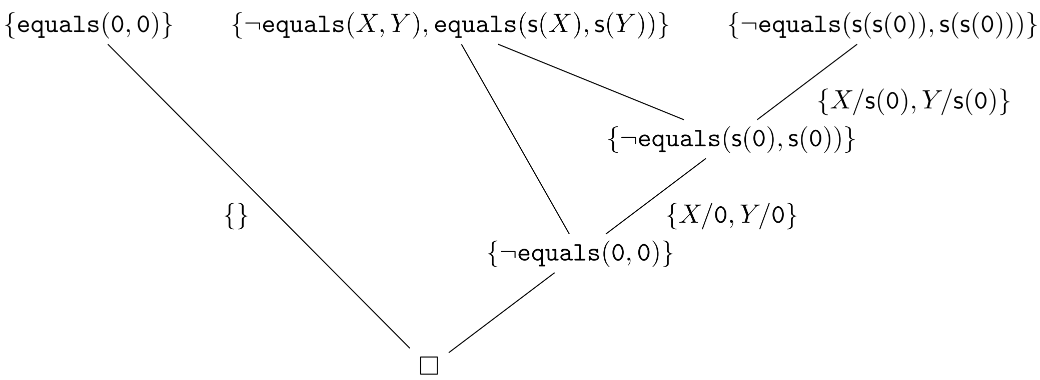
Following the solution from Exercise Sheet 3, we know that the corresponding formula in Skolem normal form is $\forall X, Y\, \psi$ with

$$\psi = \mathtt{equals}(0,0) \wedge (\mathtt{equals}(X,Y) \rightarrow \mathtt{equals}(\mathtt{s}(X),\mathtt{s}(Y))) \wedge \neg\mathtt{equals}(\mathtt{s}(\mathtt{s}(0)),\mathtt{s}(\mathtt{s}(0)))$$

Now we transform this formula into CNF and obtain the clauses:

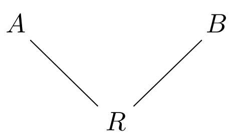
$$\begin{aligned} K_1 &= \{\mathtt{equals}(0,0)\} \\ K_2 &= \{\neg\mathtt{equals}(X,Y), \mathtt{equals}(\mathtt{s}(X),\mathtt{s}(Y))\} \\ K_3 &= \{\neg\mathtt{equals}(\mathtt{s}(\mathtt{s}(0)),\mathtt{s}(\mathtt{s}(0)))\} \end{aligned}$$

By deriving the empty clause as shown below, we prove that $\{\varphi_1, \varphi_2\} \models \varphi$ holds.

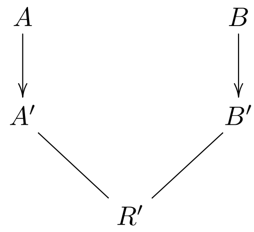


Exercise 2 (Lifting Lemma): (3 points)

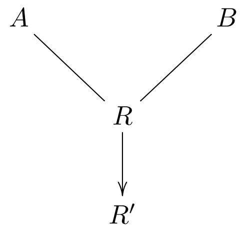
Consider the clauses $\underbrace{\{\mathtt{inc}(X,\mathtt{s}(X))\}}_{=:A}, \underbrace{\{\neg\mathtt{inc}(Y,Z), \mathtt{inc}(\mathtt{s}(Y),\mathtt{s}(Z))\}}_{=:B}$. These clauses can be resolved to $R := \{\mathtt{inc}(\mathtt{s}(X),\mathtt{s}(\mathtt{s}(X)))\}$ as follows:



For this resolution step, find all ground instances A', B', and R' of A, B, and R (using substitution with ground terms built from the function symbols s and $\mathsf{0}$), such that we have



(i.e., R' is a resolvent of A' and B') and by the lifting lemma (Lemma 3.4.8) we get:



If there is an infinite number of such ground instances for A, B, and R, give a suitable finite description of these ground instances.

Solution:

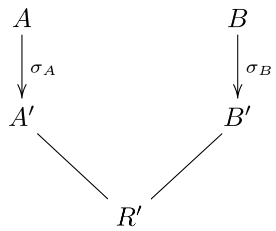
There is an infinite number of such ground instances A', B', and R'. For each $n \in \mathbb{N}_0$ by applying the substitutions

$$\begin{aligned}\sigma_A :=& \{X/\mathsf{s}^n(\mathsf{0})\}\\ \sigma_B :=& \{Y/\mathsf{s}^n(\mathsf{0}), Z/\mathsf{s}(\mathsf{s}^n(\mathsf{0}))\}\\ \sigma_R :=& \{X/\mathsf{s}^n(\mathsf{0})\}\end{aligned}$$

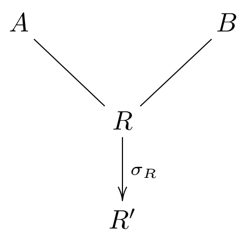
as follows

$$\begin{aligned}A' = \sigma_A(A) =& \{\mathtt{inc}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}(\mathsf{s}^n(\mathsf{0})))\}\\ B' = \sigma_B(B) =& \{\neg\mathtt{inc}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}(\mathsf{s}^n(\mathsf{0}))), \mathtt{inc}(\mathsf{s}(\mathsf{s}^n(\mathsf{0})), \mathsf{s}(\mathsf{s}(\mathsf{s}^n(\mathsf{0}))))\}\\ R' = \sigma_R(R) =& \{\mathtt{inc}(\mathsf{s}(\mathsf{s}^n(\mathsf{0})), \mathsf{s}(\mathsf{s}(\mathsf{s}^n(\mathsf{0}))))\}\end{aligned}$$

we have



Using the lifting lemma we then get



Exercise 3 (Restrictions of Resolution): (2 + 3 + 3 + 2 + 2 + 1 = 13 points)

Consider the set of clauses

$$\mathcal{K} = \{\{\mathsf{p}(\mathsf{f}(\mathsf{a}), X)\}, \{\mathsf{p}(Y, \mathsf{f}(Z)), \neg\mathsf{p}(\mathsf{f}(Y), Z)\}, \{\neg\mathsf{p}(\mathsf{a}, X), \mathsf{q}(\mathsf{f}(X), X)\}, \{\neg\mathsf{p}(\mathsf{a}, Z), \neg\mathsf{q}(\mathsf{f}(\mathsf{f}(Y)), Z), \neg\mathsf{q}(\mathsf{f}(Z), \mathsf{f}(Y))\}\}$$

with $\mathsf{a} \in \Sigma_0$, $\mathsf{f} \in \Sigma_1$, $\mathsf{q} \in \Delta_2$, and $\mathsf{p} \in \Delta_2$.

a) Derive the empty clause from $\mathcal{K}$ using full but not linear resolution (i.e., there must be at least one non-linear resolution step). For each step denote the substitutions used.

b) Derive the empty clause from $\mathcal{K}$ using linear but not input resolution. For each step denote the substitutions used.

c) Derive the empty clause from $\mathcal{K}$ using input resolution but not SLD resolution. For each step denote the substitutions used.

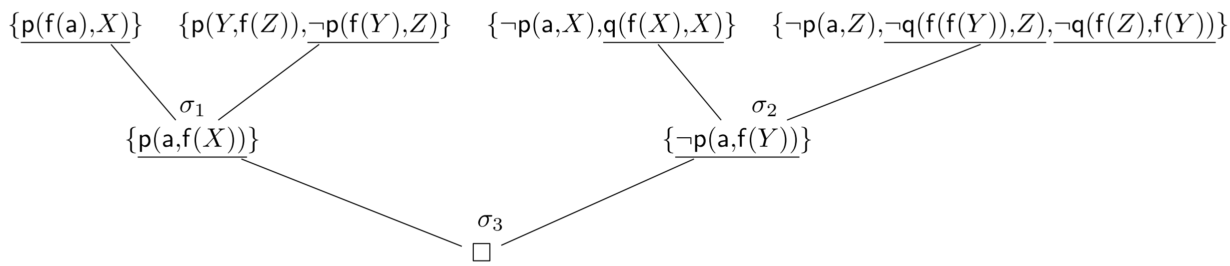
d) Derive the empty clause from $\mathcal{K}$ using SLD resolution but not binary SLD resolution. For each step denote the substitutions used. In addition, also give the answer substitution. Here, the answer substitution is computed as follows. Consider an SLD resolution proof from a negative clause N of the form $N, R_1, R_2, \ldots, R_m$, where R_m is the empty clause $\Box$ and where no variable renamings have been applied to $N, R_1, R_2, \ldots, R_m$ during the resolution proof (i.e., variable renamings are only applied to definite Horn clauses. Here, we use variable renamings to ensure that the variables in the definite parent clause are disjoint from all variables occurring in clauses that have already been used in the resolution proof.) So R_1 is the resolvent of N and a definite clause from the input set. Similarly, R_2 is the resolvent of R_1 and a definite clause from the input set, etc. In the first resolution step, let σ_1 be the used mgu. In the second resolution step, one used the mgu σ_2, etc. Then the answer substitution is $\sigma_m \circ \ldots \circ \sigma_2 \circ \sigma_1$, i.e., σ_1 is applied first in this composition of substitutions. Moreover, the answer substitution is restricted to those variables that occur in the original negative clause N.

e) Derive the empty clause from $\mathcal{K}$ using binary SLD resolution. For each step denote the substitutions used. In addition, also give the answer substitution.

f) Express $\mathcal{K}$ as queries, facts, and rules of a logic program.

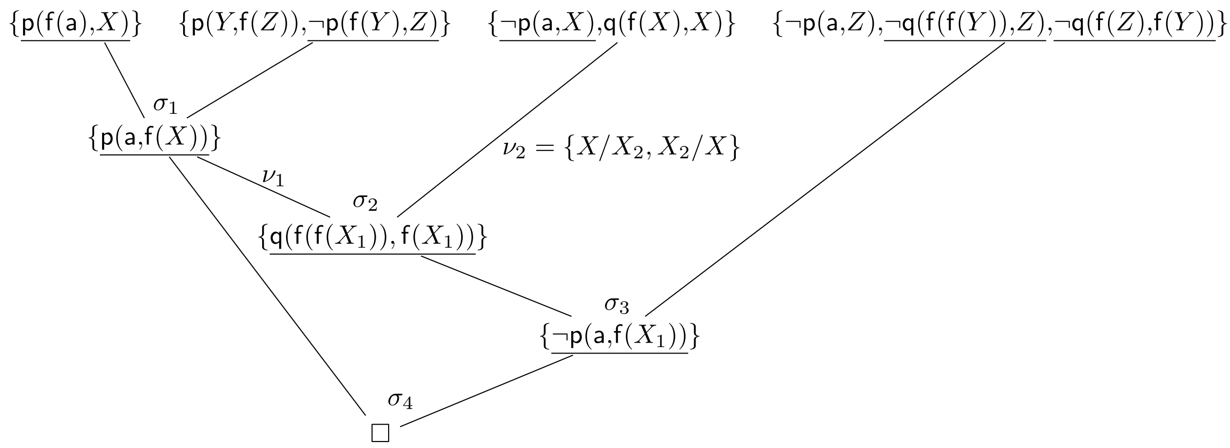
Solution:

a)



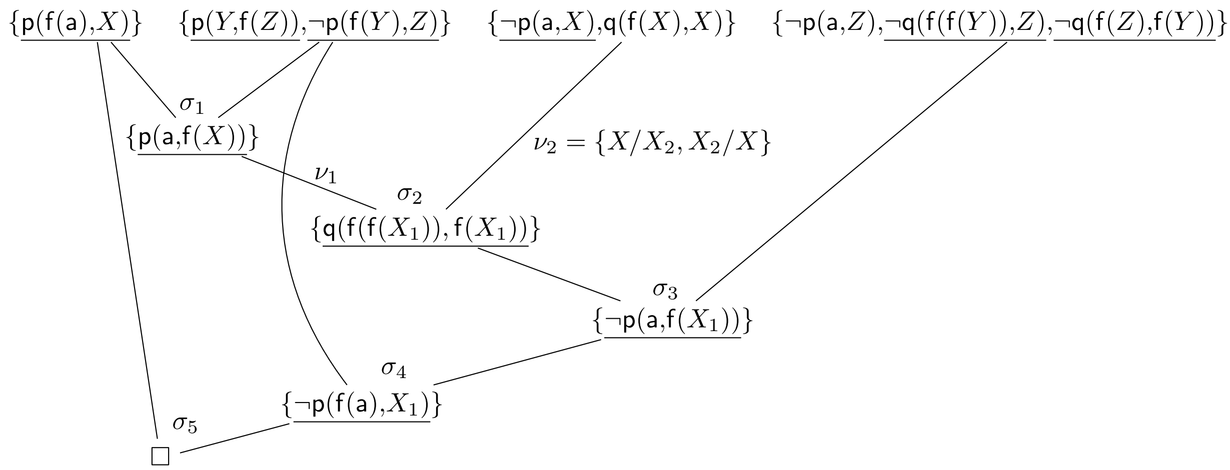
$\sigma_1 = \{Y/\mathsf{a}, Z/X\}$
$\sigma_2 = \{X/\mathsf{f}(Y), Z/\mathsf{f}(Y)\}$
$\sigma_3 = \{X/Y\}$

b)



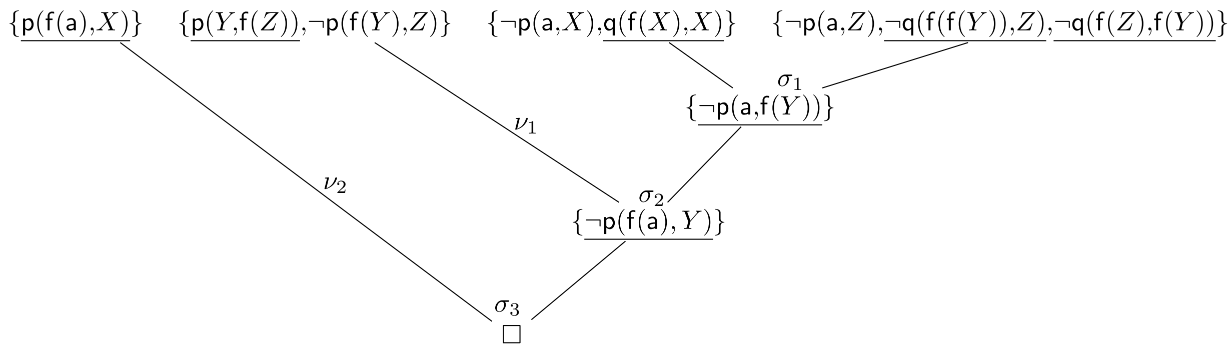
$\sigma_1 = \{Y/\mathsf{a}, Z/X\}$
$\nu_1 = \{X/X_1, X_1/X\}$
$\sigma_2 = \{X_2/\mathsf{f}(X_1)\}$
$\sigma_3 = \{Y/X_1, Z/\mathsf{f}(X_1)\}$
$\sigma_4 = \{X/X_1\}$

c)



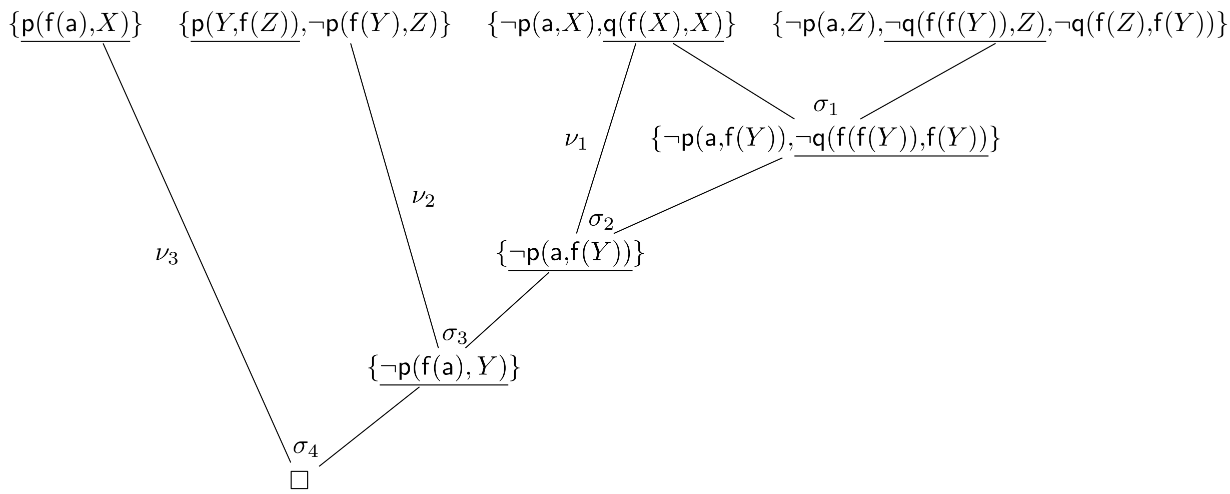
$\sigma_1 = \{Y/\mathsf{a}, Z/X\}$
$\nu_1 = \{X/X_1, X_1/X\}$
$\sigma_2 = \{X_2/\mathsf{f}(X_1)\}$
$\sigma_3 = \{Y/X_1, Z/\mathsf{f}(X_1)\}$
$\sigma_4 = \{Y/\mathsf{a}, Z/X_1\}$
$\sigma_5 = \{X/X_1\}$

d)



$\sigma_1 = \{X/\mathsf{f}(Y), Z/\mathsf{f}(Y)\}$
$\nu_1 = \{Y/Y_1, Y_1/Y, Z/Z_1, Z_1/Z\}$
$\sigma_2 = \{Y_1/\mathsf{a}, Z_1/Y\}$
$\nu_2 = \{X/X_1, X_1/X\}$
$\sigma_3 = \{X_1/Y\}$
Answer substitution: $\{Z/\mathsf{f}(Y)\}$

e)



$\sigma_1 = \{X/\mathsf{f}(Y), Z/\mathsf{f}(Y)\}$
$\nu_1 = \{X/X_1, X_1/X\}$
$\sigma_2 = \{X_1/\mathsf{f}(Y)\}$
$\nu_2 = \{Y/Y_1, Y_1/Y, Z/Z_1, Z_1/Z\}$
$\sigma_3 = \{Y_1/\mathsf{a}, Z_1/Y\}$
$\nu_3 = \{X/X_2, X_2/X\}$
$\sigma_4 = \{X_2/Y\}$
Answer substitution: $\{Z/\mathsf{f}(Y)\}$

f)

```
p(f(a),X).
p(Y,f(Z)) :- p(f(Y),Z).
q(f(X),X) :- p(a,X).

?- p(a,Z), q(f(f(Y)),Z), q(f(Z),f(Y)).
```

Exercise 4 (Unrenamed Resolution): (4 points)

A clause R is an *unrenamed resolvent* of two clauses K_1 and K_2 iff the following two conditions are satisfied:

- There are literals $L_1, \ldots, L_m \in K_1$ and $L'_1, \ldots, L'_n \in K_2$ with $m, n \geq 1$ such that $\{\overline{L_1}, \ldots, \overline{L_m}, L'_1, \ldots, L'_n\}$ is unifiable with some mgu σ.
- $R = \sigma((K_1 \setminus \{L_1, \ldots, L_m\}) \cup (K_2 \setminus \{L'_1, \ldots, L'_n\}))$

Unrenamed resolution is, thus, defined like resolution in predicate logic, but without renaming the clauses first such that they do not have any variables in common.
Please prove or disprove the following statements:

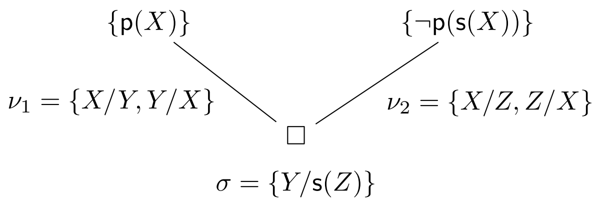
a) Unrenamed resolution is *sound*, i.e., there is no satisfiable clause set $\mathcal{K}$ from which one can derive $\Box$ by unrenamed resolution.

b) Unrenamed resolution is *complete*, i.e., from any unsatisfiable clause set $\mathcal{K}$ one can derive $\Box$ by unrenamed resolution.

Solution:

a) Any step in unrenamed resolution can be simulated by a resolution step in predicate logic by unifying all variables that had the same name before they were renamed. Therefore, soundness of unrenamed resolution follows from soundness of resolution in predicate logic (Thm. 3.4.10).

b) Completeness of unrenamed resolution is refuted by the following counter-example: Consider the clause set $\{\{\mathsf{p}(X)\}, \{\neg\mathsf{p}(\mathsf{s}(X))\}\}$ over the signature (Σ, Δ) with $\Sigma_1 = \{\mathsf{s}\}, \Sigma_0 = \{\mathsf{0}\}$, and $\Delta_1 = \{\mathsf{p}\}$. With resolution in predicate logic one can easily derive the empty clause as follows.



However, with unrenamed resolution there is no derivation at all since $\mathsf{p}(X)$ and $\mathsf{p}(\mathsf{s}(X))$ do not unify.