Exercise 1 (Semantics): (3 + 3 + 3 + 1 = 10 points)

Consider the following logic program $\mathcal{P}$ over the signature (Σ, Δ) with $\texttt{0}, \texttt{s} \in \Sigma$ and $\texttt{minus}, \texttt{pred} \in \Delta$:

```
minus(X, 0, X).
minus(X, Y, Z) :- minus(A, B, Z), pred(X, A), pred(Y, B).
pred(s(s(X)), s(Y)) :- pred(s(X), Y).
pred(s(X), X).
```

Also consider the following query:

```
?- minus(s(s(s(0))), Y, s(0)).
```

Hints:

- In each of the computation steps that you give in this exercise, you can abbreviate those parts of the substitution that did not change by „...“.

a) Show a successful computation for the query above (i.e., a computation of the form $(G, \emptyset) \vdash^+_{\mathcal{P}} (\Box, \sigma)$ where $G = \{\neg \texttt{minus}(\texttt{s}^3(\texttt{0}), Y, \texttt{s}(\texttt{0}))\}$). Also give the answer substitution.

b) Show a finite unsuccessful computation for the query above (i.e., a computation of the form $(G, \emptyset) \vdash^+_{\mathcal{P}} (G_1, \sigma_1)$ where $G = \{\neg \texttt{minus}(\texttt{s}^3(\texttt{0}), Y, \texttt{s}(\texttt{0}))\}$, $G_1 \neq \Box$ and there are no G_2 and σ_2 such that $(G_1, \sigma_1) \vdash_{\mathcal{P}} (G_2, \sigma_2)$).

c) Indicate an infinite computation for the query above by giving the first few steps. Give enough steps so that it is obvious how the infinite computation looks like.

d) What is $D[\![\mathcal{P}, G]\!]$ in this example?

Example: The query

```
?- pred(s(s(0)), Y).
```

has two successful derivations. Here we use variable renamings to replace the variables X and Y in the program clauses by X_1 and Y_1 resp. X_2:

$$(\{\underline{\neg \texttt{pred}(\texttt{s}(\texttt{s}(\texttt{0})), Y)}\}, \emptyset) \vdash_{\mathcal{P}} (\{\underline{\neg \texttt{pred}(\texttt{s}(\texttt{0}), Y_1)}\}, \{X_1/\texttt{0}, Y/\texttt{s}(Y_1)\}) \quad (1)$$
$$\vdash_{\mathcal{P}} (\Box, \{X_1/\texttt{0}, Y/\texttt{s}(\texttt{0}), X_2/\texttt{0}, Y_1/\texttt{0}\})$$

$$(\{\underline{\neg \texttt{pred}(\texttt{s}(\texttt{s}(\texttt{0})), Y)}\}, \emptyset) \vdash_{\mathcal{P}} (\Box, \{X_1/\texttt{s}(\texttt{0}), Y/\texttt{s}(\texttt{0})\}) \quad (2)$$

The answer substitution for both derivations is $\{Y/\texttt{s}(\texttt{0})\}$.

Solution:

a)

$$
\begin{array}{lr}
 & (\{\underline{\neg\mathtt{minus}(\mathtt{s}^3(\mathtt{0}), Y, \mathtt{s}(\mathtt{0}))}\}, \emptyset) \\
\vdash_{\mathcal{P}} & (\{\neg\mathtt{minus}(A_1, B_1, \mathtt{s}(\mathtt{0})), \underline{\neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), A_1)}, \neg\mathtt{pred}(Y, B_1)\}, \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(\mathtt{s}^2(\mathtt{0}), B_1, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(Y, B_1)\}, \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0}), X_2/\mathtt{s}^2(\mathtt{0}), A_1/\mathtt{s}^2(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\neg\mathtt{minus}(A_3, B_3, \mathtt{s}(\mathtt{0})), \underline{\neg\mathtt{pred}(\mathtt{s}^2(\mathtt{0}), A_3)}, \neg\mathtt{pred}(B_1, B_3), \neg\mathtt{pred}(Y, B_1)\}, \{\ldots, X_3/\mathtt{s}^2(\mathtt{0}), Y_3/B_1, Z_3/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(\mathtt{s}(\mathtt{0}), B_3, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(B_1, B_3), \neg\mathtt{pred}(Y, B_1)\}, \{\ldots, X_4/\mathtt{s}(\mathtt{0}), A_3/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{pred}(B_1, \mathtt{0})}, \neg\mathtt{pred}(Y, B_1)\}, \{\ldots, X_5/\mathtt{s}(\mathtt{0}), B_3/\mathtt{0}\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{pred}(Y, \mathtt{s}(\mathtt{0}))}\}, \{\ldots, X_6/\mathtt{0}, B_1/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\Box, \{\ldots, X_7/\mathtt{s}(\mathtt{0}), Y/\mathtt{s}^2(\mathtt{0})\})
\end{array}
$$

The answer substitution is $\{Y/\mathtt{s}^2(\mathtt{0})\}$.

b)

$$
\begin{array}{lr}
 & (\{\underline{\neg\mathtt{minus}(\mathtt{s}^3(\mathtt{0}), Y, \mathtt{s}(\mathtt{0}))}\}, \emptyset) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(A_1, B_1, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), A_1), \neg\mathtt{pred}(Y, B_1)\}, \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), \mathtt{s}(\mathtt{0})), \underline{\neg\mathtt{pred}(Y, \mathtt{0})}\}, \{\ldots, X_2/\mathtt{s}(\mathtt{0}), A_1/\mathtt{s}(\mathtt{0}), B_1/\mathtt{0}\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), \mathtt{s}(\mathtt{0}))}\}, \{\ldots, X_3/\mathtt{0}, Y/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\neg\mathtt{pred}(\mathtt{s}^2(\mathtt{0}), \mathtt{0})\}, \{\ldots, X_4/\mathtt{s}(\mathtt{0}), Y_4/\mathtt{0}\})
\end{array}
$$

c)

$$
\begin{array}{lr}
 & (\{\underline{\neg\mathtt{minus}(\mathtt{s}^3(\mathtt{0}), Y, \mathtt{s}(\mathtt{0}))}\}, \emptyset) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(A_1, B_1, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), A_1), \neg\mathtt{pred}(Y, B_1)\}, \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(A_2, B_2, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(A_1, A_2), \neg\mathtt{pred}(B_1, B_2), \neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), A_1), \neg\mathtt{pred}(Y, B_1)\}, \\
 & \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0}), X_2/A_1, Y_2/B_1, Z_2/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & (\{\underline{\neg\mathtt{minus}(A_3, B_3, \mathtt{s}(\mathtt{0}))}, \neg\mathtt{pred}(A_2, A_3), \neg\mathtt{pred}(B_2, B_3), \\
 & \neg\mathtt{pred}(A_1, A_2), \neg\mathtt{pred}(B_1, B_2), \neg\mathtt{pred}(\mathtt{s}^3(\mathtt{0}), A_1), \neg\mathtt{pred}(Y, B_1)\}, \\
 & \{X_1/\mathtt{s}^3(\mathtt{0}), Y_1/Y, Z_1/\mathtt{s}(\mathtt{0}), X_2/A_1, Y_2/B_1, Z_2/\mathtt{s}(\mathtt{0}), X_3/A_2, Y_3/B_2, Z_3/\mathtt{s}(\mathtt{0})\}) \\
\vdash_{\mathcal{P}} & \ldots
\end{array}
$$

d) $D[\![\mathcal{P}, G]\!] = \{\mathtt{minus}(\mathtt{s}^3(\mathtt{0}), \mathtt{s}^2(\mathtt{0}), \mathtt{s}(\mathtt{0}))\}$.

Exercise 2 (Fixpoints): (3 + 3 + 3 = 9 points)

Consider the function $f\colon \mathit{Pot}(\mathbb{N}) \to \mathit{Pot}(\mathbb{N})$.

$$
f(M) = \begin{cases} \{0\}, & \text{if } M = \emptyset \\ \{(\sum\limits_{y \in X} y) - \min X \mid \emptyset \neq X \subseteq M\}, & \text{if } M \neq \emptyset \text{ is finite} \\ \mathbb{N}, & \text{otherwise} \end{cases}
$$

So for example, $f(\{2,5\}) = \{(\sum\limits_{y \in \{2\}} y) - \min\{2\}, (\sum\limits_{y \in \{5\}} y) - \min\{5\}, (\sum\limits_{y \in \{2,5\}} y) - \min\{2,5\}\} = \{0, 5\}$.

a) Prove that f is monotonic.

b) Prove or disprove that f is continuous.

c) Please give all fixpoints of f and mark the least fixpoint.

Solution:

a) Consider two sets M_1, M_2 with $M_1 \subseteq M_2 \subseteq \mathbb{N}$. We have to show $f(M_1) \subseteq f(M_2)$.

- Assume $M_1 = \emptyset$ and M_2 is finite. We then have $f(M_1) = \{0\}$. If $M_2 = \emptyset$, we have $f(M_1) = f(M_2) = \{0\}$. Otherwise, M_2 contains at least one element n, so if we choose $X = \{n\}$, we get $(\sum_{y \in \{n\}} y) - \min\{n\} = n - n = 0 \in f(M_2)$. Therefore, $f(M_1) \subseteq f(M_2)$ holds.
- We cannot have that $M_1 \neq \emptyset$ and $M_2 = \emptyset$ because $M_1 \subseteq M_2$ holds.
- Assume both M_1 and M_2 are finite and $M_1 \neq \emptyset \neq M_2$. We then have

$$f(M_1) = \{(\sum_{y \in X} y) - \min X \mid \emptyset \neq X \subseteq M_1\}$$

  and

$$f(M_2) = \{(\sum_{y \in X} y) - \min X \mid \emptyset \neq X \subseteq M_2\}$$

  Let $n \in f(M_1)$. Then there exists an $X \subseteq M_1$ such that $n = (\sum_{y \in X} y) - \min X$. Then we also have $X \subseteq M_2$ and thus, $n = (\sum_{y \in X} y) - \min X \in f(M_2)$.
- Assume M_2 is infinite. We then have $f(M_2) = \mathbb{N}$, so obviously $f(M_1) \subseteq f(M_2)$ holds.
- We cannot have that M_1 is infinite and M_2 is finite because $M_1 \subseteq M_2$ holds.

b) Consider the chain $(M_i)_{i \in \mathbb{N}}$ with $M_n = \{i \mid 1 \leq i \leq n\}$ ($\emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \ldots$). With this we have $f(M_0) = \{0\} \subseteq f(M_1) = \{0\} \subseteq f(M_2) = \{0,2\} \subseteq f(M_3) = \{0,2,3,5\} \subseteq f(M_4) = \{0,2,3,4,5,6,7,9\} \subseteq \ldots$. We see that $\bigcup_{i \in \mathbb{N}} M_i = \mathbb{N} \setminus \{0\}$ and $\bigcup_{i \in \mathbb{N}} f(M_i) = \mathbb{N} \setminus \{1\}$. Now we see that f is not continuous:

$$\bigcup_{i \in \mathbb{N}} f(M_i) = \mathbb{N} \setminus \{1\} \subsetneq f(\bigcup_{i \in \mathbb{N}} M_i) = f(\mathbb{N} \setminus \{0\}) = \mathbb{N}$$

c)
- $\{0\}$
- $\{0, n\}$ for all $n \in \mathbb{N}, n \geq 1$
- $\mathbb{N}$

The least fixpoint is $\{0\}$ (because it is the smallest of all fixpoints wrt. set inclusion, i.e. $\{0\} \subseteq \{0, n\}$ for all n and $\{0\} \subseteq \mathbb{N}$).

Exercise 3 (Fixpoint Semantics): (3 + 1 + 1 + 1 = 6 points)

Reconsider the logic program $\mathcal{P}$ over the signature (Σ, Δ) with $\mathtt{0}, \mathtt{s} \in \Sigma$ and $\mathtt{minus}, \mathtt{pred} \in \Delta$ from Exercise 1, where we remove the recursive rule of the predicate pred:

```
minus(X, 0, X).
minus(X, Y, Z) :- minus(A, B, Z), pred(X, A), pred(Y, B).
pred(s(X), X).
```

a) For each $i \in \mathbb{N}$ explicitly give $\underline{\text{trans}}_{\mathcal{P}}^i(\emptyset)$.

b) Compute the set $\text{lfp}(\underline{\text{trans}}_{\mathcal{P}})$.

c) Give $F[\![\mathcal{P}, \{\neg\texttt{minus(s(s(0)), Y, Z)}\}]\!]$.

d) Give $F[\![\mathcal{P}, \{\neg\texttt{minus(X, s(Y), Y)}\}]\!]$.

Solution:

As usual, $\mathcal{T}(\Sigma)$ is the set of all ground terms $\{\texttt{0}, \texttt{s(0)}, \texttt{s(s(0))}, \ldots\}$.

a)

$$\begin{aligned}
\underline{\text{trans}}^0_{\mathcal{P}}(\emptyset) &= \emptyset \\
\underline{\text{trans}}^1_{\mathcal{P}}(\emptyset) &= \{\texttt{minus}(t, \texttt{0}, t), \texttt{pred}(\texttt{s}(t), t) \mid t \in \mathcal{T}(\Sigma)\} \\
\underline{\text{trans}}^2_{\mathcal{P}}(\emptyset) &= \{\texttt{minus}(\texttt{s}(t), \texttt{s(0)}, t) \mid t \in \mathcal{T}(\Sigma)\} \cup \underline{\text{trans}}^1_{\mathcal{P}}(\emptyset) \\
\underline{\text{trans}}^3_{\mathcal{P}}(\emptyset) &= \{\texttt{minus}(\texttt{s}^2(t), \texttt{s}^2(\texttt{0}), t) \mid t \in \mathcal{T}(\Sigma)\} \cup \underline{\text{trans}}^2_{\mathcal{P}}(\emptyset) \\
&\vdots \\
\underline{\text{trans}}^{i+1}_{\mathcal{P}}(\emptyset) &= \{\texttt{minus}(\texttt{s}^j(t), \texttt{s}^j(\texttt{0}), t), \texttt{pred}(\texttt{s}(t), t) \mid t \in \mathcal{T}(\Sigma), 0 \leq j \leq i\}
\end{aligned}$$

b) $\text{lfp}(\underline{\text{trans}}_{\mathcal{P}}) = \{\texttt{minus}(\texttt{s}^i(t), \texttt{s}^i(\texttt{0}), t), \texttt{pred}(\texttt{s}(t), t) \mid t \in \mathcal{T}(\Sigma), i \in \mathbb{N}\}$

c) $F[\![\mathcal{P}, \{\neg\texttt{minus(s(s(0)), Y, Z)}\}]\!] = \{\texttt{minus}(\texttt{s}^2(\texttt{0}), \texttt{0}, \texttt{s}^2(\texttt{0})), \texttt{minus}(\texttt{s}^2(\texttt{0}), \texttt{s(0)}, \texttt{s(0)}), \texttt{minus}(\texttt{s}^2(\texttt{0}), \texttt{s}^2(\texttt{0}), \texttt{0})\}$

d) $F[\![\mathcal{P}, \{\neg\texttt{minus(X, s(Y), Y)}\}]\!] = \{\texttt{minus}(\texttt{s}^{2i+1}(\texttt{0}), \texttt{s}^{i+1}(\texttt{0}), s^i(\texttt{0})) \mid i \in \mathbb{N}\}$

Exercise 4 (Proofs): (2 + 2 = 4 points)

a) Please show that every continuous function $f \colon \mathcal{P}ot(M) \to \mathcal{P}ot(M)$ is monotonic.

b) Please show that for every *finite* chain

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$$

and every monotonic function $f \colon \mathcal{P}ot(M) \to \mathcal{P}ot(M)$ we have:

$$f(\bigcup_{i=1}^{n} M_i) = \bigcup_{i=1}^{n} f(M_i)$$

Solution:

a) Let f be a continuous function. Then for every chain $M_1 \subseteq M_2 \subseteq \ldots$ we have:

$$f(\bigcup_{i \in \mathbb{N}} M_i) = \bigcup_{i \in \mathbb{N}} f(M_i)$$

Let A, B be sets with $A \subseteq B$. Our goal is to show that $f(A) \subseteq f(B)$. Because of $A \subseteq B$, we know that A and B form a chain. It follows that $f(A \cup B) = f(A) \cup f(B)$. Because of $A \subseteq B$ we know $f(B) = f(A \cup B) = f(A) \cup f(B)$, and finally $f(A) \subseteq f(B)$.

b) Let f be a monotonic function and let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n$ be a finite chain. Because f is monotonic, from

$$M_i \subseteq M_n \text{ for all } i \in \{1, \dots, n\}$$

it follows that

$$f(M_i) \subseteq f(M_n) \text{ for all } i \in \{1, \dots, n\}$$

So we have:

$$\bigcup_{i=1}^{n} f(M_i) = f(M_n) = f(\bigcup_{i=1}^{n} M_i)$$