Semantics by lub's of Approximations

```
fact :: Int -> Int
fact  = \x -> if x <= 0 then 1 else fact(x-1) * x
```

Regard non-recursively defined approximations

$$\mathtt{fact}_0 = \backslash\mathtt{x} \texttt{ -> bot}$$
$$\mathtt{fact}_1 = \backslash\mathtt{x} \texttt{ -> if x <= 0 then 1 else } \mathtt{fact}_0(\mathtt{x}-1) * \mathtt{x}$$
$$\mathtt{fact}_2 = \backslash\mathtt{x} \texttt{ -> if x <= 0 then 1 else } \mathtt{fact}_1(\mathtt{x}-1) * \mathtt{x}$$
$$\ldots$$

Thus: $\mathtt{fact}_{i+1} = \mathtt{ff}\ \mathtt{fact}_i$ where

```
ff :: (Int -> Int) -> (Int -> Int)
ff g  = \x -> if x <= 0 then 1 else g(x-1) * x
```

Define semantics fact of `fact` as

$$\mathit{fact} = \sqcup\{\mathit{fact}_0, \mathit{fact}_1, \ldots\} = \sqcup\{\mathit{ff}^{\,i}(\bot) \mid i \in \mathbb{N}\}$$

Semantics by Least Fixpoints

```
fact :: Int -> Int
fact  = \x -> if x <= 0 then 1 else fact(x-1) * x
```

Semantics fact of `fact` should *satisfy* the defining equation

$$\mathit{fact} = \underbrace{\texttt{\textbackslash x -> if x <= 0 then 1 else}\ \mathit{fact}(\mathtt{x}-1) * \mathtt{x}}_{\mathit{ff}\,(\mathit{fact})}$$

where

```
ff :: (Int -> Int) -> (Int -> Int)
ff g  = \x -> if x <= 0 then 1 else g(x-1) * x
```

fact should be undefined unless program enforces its definedness.
Define semantics fact of `fact` as

$$\mathit{fact} = \mathrm{lfp}\ \mathit{ff}$$

Fixpoint Theorem

Both semantics are equivalent:

$$\mathrm{lfp}\ \mathit{ff} \quad = \quad \sqcup\{\mathit{ff}^{\,i}(\bot) \mid i \in \mathbb{N}\}$$

Thm. 2.1.17 (*Fixpoint Theorem*)
Let $\sqsubseteq$ be a cpo on D and let $f : D \to D$ be continuous.
Then f has a least fixpoint and we have $\mathrm{lfp}\, f \ = \ \sqcup\{f^i(\bot)|i \in \mathbb{N}\}$.

Proof.
$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq f^3(\bot) \sqsubseteq \ldots$ by monotonicity of f

So $\{f^i(\bot)|i \in \mathbb{N}\}$ is a chain
and $\sqcup\{f^i(\bot)|i \in \mathbb{N}\}$ exists by completeness of $\sqsubseteq$.

To show: $\sqcup\{f^i(\bot)|i \in \mathbb{N}\}$ is the least fixpoint of f