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Exercise 1 (Theoretical Foundations): (6 × 4 = 24 points)

Give a short proof sketch or a counterexample for each of the following statements:

a) The question whether $s \equiv_{\mathcal{E}} t$ holds is semi-decidable for every set $\mathcal{E}$ of equations between terms.

b) For every terminating term rewrite system $\mathcal{R}$, $\to_{\mathcal{R}}^{+}$ is a simplification order.

c) For every set M and every relation $\to \subseteq M \times M$ such that every $p \in M$ has at most one $\to$-normal form, we have that $\to$ is confluent.

d) Let Σ be a signature with $\mathsf{c} \in \Sigma_2$, let x, y be variables. If $\{\mathsf{c}(x, y) \equiv x, \mathsf{c}(x, y) \equiv y\} \subseteq \mathcal{E}$, then for all terms s, t we have $s \equiv_{\mathcal{E}} t$.

e) The embedding order on terms is normalizing.

f) A relation $\to$ is called *strongly confluent* iff for all p, s, and t with $p \to s$ and $p \to t$ there is a q with $s \to^{=} q$ and $t \to^{=} q$. Here, $s \to^{=} q$ means that $s \to q$ or $s = q$ holds. Then every confluent relation is also strongly confluent.

Solution:

a) True. One can build a search tree starting in s by applying the equations from $\mathcal{E}$. As the tree can be of infinite breadth if $\mathcal{V}(l) \neq \mathcal{V}(r)$ for some $l \equiv r \in \mathcal{E}$, a suitable (diagonal) expansion strategy has to be used.

b) False. The system $\mathcal{R}$ containing only the rule $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$ is terminating. However $\to_{\mathcal{R}}^{+}$ is no simplification order, since for every simplification order $\succ$, we have $\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \succ \mathsf{f}(\mathsf{f}(x))$, which in turn would contradict the irreflexivity required of simplification orders.

c) False. Consider the relation $\to$ over the set $M = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ where only $\mathsf{a} \to \mathsf{b}, \mathsf{a} \to \mathsf{c}, \mathsf{b} \to \mathsf{b}, \mathsf{c} \to \mathsf{c}$ hold. Then each object has at most one normal form (because no object has a normal form). But we have $\mathsf{b} \leftarrow^{*} \mathsf{a} \to^{*} \mathsf{c}$, and the objects b and c are not joinable, which refutes confluence.

d) True. Let s, t be arbitrary terms. By Birkhoff's theorem, we have $s \equiv_{\mathcal{E}} t$ iff $s \leftrightarrow_{\mathcal{E}}^{*} t$. The statement then follows by $s \leftarrow_{\mathcal{E}} \mathsf{c}(s, t) \to_{\mathcal{E}} t$.

e) True. The lecture states (in Lemma 3.3.11) that every well-founded relation is normalizing, and the embedding order is well founded according to the lecture (Lemma 4.3.2).

f) False. Consider the following TRS $\mathcal{R}$.

$$
\begin{aligned}
\mathsf{a} &\to \mathsf{b} \\
\mathsf{a} &\to \mathsf{d} \\
\mathsf{b} &\to \mathsf{c} \\
\mathsf{c} &\to \mathsf{d}
\end{aligned}
$$

The relation $\to_{\mathcal{R}}$ is confluent, but not strongly confluent. Thus, the proposition is disproved.

Exercise 2 (Equivalence Classes): (8 + 12 + 4 = 24 points)

a) Let $s \sim t$ hold for two terms s and t iff $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s| = |t|$. Here, $|s|$ is the *size* of the term s where $|x| = 1$ for any variable $x \in \mathcal{V}$ and $|f(s_1, \ldots, s_n)| = 1 + |s_1| + \cdots + |s_n|$ for any function symbol $f \in \Sigma$.

Please show that $\sim$ is an equivalence relation and that all equivalence classes w.r.t. $\sim$ are finite.

Hints:

- Remember that we only consider finite signatures Σ.

b) Please show that the word problem is decidable for each set of equations $\mathcal{E}$ where $\equiv_\mathcal{E} \subseteq \sim$. To this end, describe a decision procedure which decides for arbitrary given input terms s and t and a set of equations $\mathcal{E}$ with $\equiv_\mathcal{E} \subseteq \sim$ whether $s \equiv_\mathcal{E} t$ holds.

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

c) Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{f}, \mathsf{g}, \mathcal{O}\}$.

$$\begin{aligned}
\mathsf{f}(x, y) &\equiv \mathsf{f}(y, x)\\
\mathsf{g}(x, \mathcal{O}, y, z) &\equiv \mathsf{f}(x, \mathsf{f}(y, z))\\
\mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) &\equiv \mathsf{g}(y, \mathcal{O}, \mathcal{O}, x)
\end{aligned}$$

Prove or disprove: $\mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) \equiv_\mathcal{E} \mathsf{f}(x, \mathsf{f}(\mathcal{O}, y))$.

Hints:

- You may use that $\equiv_\mathcal{E} \subseteq \sim$ holds.

Solution:

a) $\sim$ is an equivalence relation as both conditions are equalities and $=$ is an equivalence relation. Let s be an arbitrary term. Furthermore, let $k = |s|$. We now show that the equivalence class $[s]_\sim$ is finite. This implies that all equivalence classes w.r.t. $\sim$ are finite, since s is an arbitrary term. Let n be the number of function symbols in our signature and $m = |\mathcal{V}(s)|$. Then the maximal number of different terms of size k and only variables from $\mathcal{V}(s)$ is limited by $(n + m)^k$. Hence, $[s]_\sim$ is finite.

b) Since we have $\equiv_\mathcal{E} \subseteq \sim$, the equivalence classes w.r.t. $\equiv_\mathcal{E}$ are subsets of the equivalence classes w.r.t. $\sim$ and, therefore, finite. Knowing that all equivalence classes w.r.t. $\equiv_\mathcal{E}$ are finite, for two given terms s and t we can decide whether $s \equiv_\mathcal{E} t$ holds by computing the finite equivalence class $[s]_{\equiv_\mathcal{E}}$ and return yes if $t \in [s]_{\equiv_\mathcal{E}}$ and no otherwise. The computation of the equivalence class can for example be done by building the search tree starting from s and pruning this tree every time we reach a node which already occurred before. This tree must be finite, since it contains exactly the elements of the finite equivalence class $[s]_{\equiv_\mathcal{E}}$.

c) As we have $\equiv_\mathcal{E} \subseteq \sim$, we use the decision procedure from the previous part of the exercise. The search tree for $\mathsf{g}(\mathcal{O}, x, y, \mathcal{O})$ w.r.t. $\mathcal{E}$ is given below. Here, we deleted nodes that already occurred before.

$$\begin{tikzcd}
 & \mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) \arrow[d] & \\
 & \mathsf{g}(y, \mathcal{O}, \mathcal{O}, x) \arrow[d] & \\
 & \mathsf{f}(y, \mathsf{f}(\mathcal{O}, x)) \arrow[dl] \arrow[dr] & \\
\mathsf{f}(\mathsf{f}(\mathcal{O}, x), y) \arrow[d] & & \mathsf{f}(y, \mathsf{f}(x, \mathcal{O})) \arrow[d] \\
\mathsf{f}(\mathsf{f}(x, \mathcal{O}), y) & & \mathsf{g}(y, \mathcal{O}, x, \mathcal{O})
\end{tikzcd}$$

We cannot obtain any new nodes which do not already exist in the search tree. As $\mathsf{f}(x, \mathsf{f}(\mathcal{O}, y))$ does not occur in the search tree, we have shown that $\mathsf{g}(\mathcal{O}, x, y, \mathcal{O}) \not\equiv_{\mathcal{E}} \mathsf{f}(x, \mathsf{f}(\mathcal{O}, y))$.

Exercise 3 (Diamond Lemma): (10 points)

Let M be some set and $\to \subseteq M \times M$ some well-founded, locally confluent relation.
Prove the diamond lemma, i.e., that $\to$ is confluent.

Solution:

As $\to$ is well founded, we may use it as induction relation for a Noetherian induction. Let $p \to^* s, p \to^* t$. We show that there is a q with $s \to^* q$, $t \to^* q$.
The case $s = p$ is trivial, as then, $p = s \to^* t = q$ holds. The case $t = p$ is completely analogous.
Thus, we now consider $s \leftarrow^+ p \to^+ t$. Let $s', t' \in M$ such that $p \to s' \to^* s$ and $p \to t' \to^* t$. By local confluence, there is some $q' \in M$ with $s' \to^* q' \leftarrow^* t'$.
As we have $p \to s'$, we can apply the induction hypothesis on $s \leftarrow^* s' \to^* q'$, yielding some $u \in M$ with $s \to^* u$ and $q' \to^* u$.
As we also have $p \to t'$, we can apply the induction hypothesis on $t \leftarrow^* t' \to^* q' \to^* u$, yielding q with $u \to^* q$ and $t \to^* q$. Together with $s \to^* u$, we have $s \to^* q$ and $t \to^* q$, as desired.

Exercise 4 (Termination): (11 + 3 = 14 points)

Prove or disprove termination of the following term rewrite systems either by means of a reduction order or by a counterexample. In case you prove termination, explicitly state which order you used (including precedence and status if appropriate) and which checks you have to perform for the termination proof (you may do this e.g. with the proof tree notation from the homework exercises where embedding is considered as one step). Here, x, y, xs, and ys denote variables while all other identifiers denote function symbols. Intuitively, mullength multiplies the lengths of the two argument lists.

a)

$$\begin{aligned}
\mathsf{mullength}(\mathsf{Nil}, ys) &\to \mathcal{O}\\
\mathsf{mullength}(\mathsf{Cons}(x, xs), ys) &\to \mathsf{addlength}(ys, \mathsf{mullength}(ys, xs))\\
\mathsf{addlength}(\mathsf{Nil}, y) &\to y\\
\mathsf{addlength}(\mathsf{Cons}(x, xs), y) &\to \mathsf{addlength}(xs, \mathsf{s}(y))
\end{aligned}$$

b)

$$\begin{aligned}
\mathsf{g}(\mathsf{f}(\mathsf{a}), y) &\to \mathsf{a}\\
\mathsf{g}(x, \mathsf{f}(\mathsf{a})) &\to \mathsf{g}(\mathsf{f}(\mathsf{a}), x)
\end{aligned}$$

Solution:

a) The TRS is terminating. For the proof, we choose the RPOS with precedence $\mathsf{mullength} \sqsupset \mathsf{addlength} \sqsupset \mathsf{Cons} \sqsupset \mathsf{s} \sqsupset \mathsf{Nil} \sqsupset \mathcal{O}$ and the following status S:

$$\begin{aligned}
S(\mathsf{mullength}) &= mul\\
S(\mathsf{addlength}) &= \langle 1, 2\rangle\\
S(\mathsf{Cons}) &= \langle 1, 2\rangle
\end{aligned}$$

$$\dfrac{}{\mathsf{mullength}(\mathsf{Nil}, ys) \succ_{rpos} \mathcal{O}}\;2$$

$$\dfrac{\dfrac{}{\mathsf{mullength}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} ys}\;\succ_{emb} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{Cons}(x, xs) \succ_{rpos} xs}\;\succ_{emb}}{\{\mathsf{Cons}(x, xs), ys\}(\succ_{rpos})_{mul}\{ys, xs\}}\;mul}{\mathsf{mullength}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} \mathsf{mullength}(ys, xs)}\;3}{\mathsf{mullength}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} \mathsf{addlength}(ys, \mathsf{mullength}(ys, xs))}\;2$$

$$\dfrac{}{\mathsf{addlength}(\mathsf{Nil}, y) \succ_{rpos} y}\;\succ_{emb}$$

$$\dfrac{\dfrac{}{\mathsf{Cons}(x, xs) \succ_{rpos} xs}\;\succ_{emb} \qquad \dfrac{\dfrac{}{\mathsf{addlength}(\mathsf{Cons}(x, xs), y) \succ_{rpos} y}\;\succ_{emb}}{\mathsf{addlength}(\mathsf{Cons}(x, xs), y) \succ_{rpos} \mathsf{s}(y)}\;2}{\mathsf{addlength}(\mathsf{Cons}(x, xs), y) \succ_{rpos} \mathsf{addlength}(xs, \mathsf{s}(y))}\;3$$

b) The TRS is non-terminating for the term $\mathsf{g}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{a}))$ since this term can be rewritten to itself.

Exercise 5 (Confluence): (16 points)

A relation $\to$ is called *semi-confluent* iff for all p, s, and t with $p \to^* s$ and $p \to t$ there is a q with $s \to^* q$ and $t \to^* q$.
Prove that every semi-confluent relation is also confluent.

Hints:

- Use induction (but note that $\to$ is not necessarily well founded).

Solution:

We show the following equivalent proposition:
Let $\to$ be a semi-confluent relation. Then we have for all p, s, and t with $p \to^* s$ and $p \to^n t$ that there is a q with $s \to^* q$ and $t \to^* q$.
We use induction over n.
If $n = 0$, then we have $p = t$. We choose $q = s$ and obtain $t \to^* q$ and $s \to^0 q$.
If $n > 0$, there is a t' with:

$$\begin{tikzcd}
& p \arrow[dl, "*"'] \arrow[dr, "n-1"] & & \\
s & & t' \arrow[dr] & \\
& & & t
\end{tikzcd}$$

By the induction hypothesis, there is a q' with:

$$\begin{tikzcd}
& p \arrow[dl, "*"'] \arrow[dr, "n-1"] & & \\
s \arrow[dr, "*"] & & t' \arrow[dl, "*"'] \arrow[dr] & \\
& q' & & t
\end{tikzcd}$$

Since $\to$ is semi-confluent, there is a q with:

$$\begin{tikzcd}
& p \arrow[dl, "*"'] \arrow[dr, "n-1"] & & \\
s \arrow[dr, "*"] & & t' \arrow[dl, "*"'] \arrow[dr] & \\
& q' \arrow[dr, "*"'] & & t \arrow[dl, "*"] \\
& & q &
\end{tikzcd}$$

All in all, we have that $s \to^* q$ and $t \to^* q$ as desired.

Exercise 6 (Word Problem): (12 + 4 = 16 points)

Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{plus}, \mathsf{s}, \mathcal{O}\}$.

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \equiv & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \equiv & \mathsf{s}(\mathsf{plus}(x, y)) \\ \mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) & \equiv & \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))) \end{array}$$

Prove or disprove the following equations.

a) $\mathsf{plus}(x, \mathcal{O}) \equiv_{\mathcal{E}} x$

b) $\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) \equiv_{\mathcal{E}} \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), x)$

Solution:

We first construct a convergent TRS $\mathcal{R}$ which is equivalent to $\mathcal{E}$. For equivalence, we just orient the equations in $\mathcal{E}$ to obtain $\mathcal{R}$.

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \\ \mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) & \to & \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))) \end{array}$$

Now we prove that $\mathcal{R}$ is convergent. For termination, we choose the LPO with precedence $\mathsf{plus} \sqsupset \mathsf{s} \sqsupset \mathcal{O}$.

$$\dfrac{}{\mathsf{plus}(\mathcal{O}, y) \succ_{lpo} y}\succ_{emb}$$

$$\dfrac{\dfrac{}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{lpo} \mathsf{plus}(x, y)}\succ_{emb}}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{lpo} \mathsf{s}(\mathsf{plus}(x, y))}\,2$$

$$\dfrac{\dfrac{\dfrac{}{\mathsf{s}(x) \succ_{lpo} x}\succ_{emb} \qquad \dfrac{\dfrac{}{\mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) \succ_{lpo} \mathsf{plus}(y, z)}\succ_{emb}}{\mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) \succ_{lpo} \mathsf{s}(\mathsf{plus}(y, z))}\,2}{\mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) \succ_{lpo} \mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))}\,3}{\mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(\mathsf{s}(y), z)) \succ_{lpo} \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z))))}\,2$$

For confluence, it now suffices to prove local confluence, since $\mathcal{R}$ is terminating. To this end, we compute all critical pairs.

1. $\langle\ \mathsf{s}(\mathsf{plus}(x', \mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(\mathsf{s}(y), z))))),\quad \mathsf{plus}(\mathsf{s}(x'), \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))))\ \rangle$
2. $\langle\ \mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(\mathsf{s}(y), z))),\quad \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z))))\ \rangle$
3. $\langle\ \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))),\quad \mathsf{plus}(\mathsf{s}(x), \mathsf{s}(\mathsf{plus}(y, z)))\ \rangle$

For the first critical pair, we obtain:

$$\begin{tikzcd}[column sep=tiny] \mathsf{s}(\mathsf{plus}(x', \mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(\mathsf{s}(y), z))))) \arrow[dr, "\mathcal{R}"] & & \mathsf{plus}(\mathsf{s}(x'), \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z))))) \arrow[dl, "\mathcal{R}"'] \\ & \mathsf{s}(\mathsf{plus}(x', \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))))) & \end{tikzcd}$$

For the second critical pair, we obtain:

$$\mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(\mathsf{s}(y), z))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z))))$$

For the third critical pair, we obtain:

$$\mathsf{s}(\mathsf{plus}(x, \mathsf{s}(\mathsf{plus}(y, z)))) \leftarrow_{\mathcal{R}} \mathsf{plus}(\mathsf{s}(x), \mathsf{s}(\mathsf{plus}(y, z)))$$

Thus, all critical pairs have a common reduct and $\mathcal{R}$ is locally confluent. Because of its termination, it is also confluent and hence convergent.
With the convergent TRS $\mathcal{R}$ being equivalent to $\mathcal{E}$, we can decide the word problem for $\mathcal{E}$ by reducing both sides of an equation to normal forms w.r.t. $\mathcal{R}$ and checking them for equality.

a) Both $\mathsf{plus}(x, \mathcal{O})$ and x are normal forms w.r.t. $\mathcal{R}$ and not equal. Thus, we have $\mathsf{plus}(x, \mathcal{O}) \not\equiv_{\mathcal{E}} x$.

b) We obtain

$$\begin{array}{rcl} \mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{plus}(\mathsf{s}(\mathcal{O}), x))) \\ & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) \\ & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, x))) \\ & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{s}(x)) \end{array}$$

and

$$\begin{array}{rcl} \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), x) & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) \\ & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, x))) \\ & \to_{\mathcal{R}} & \mathsf{s}(\mathsf{s}(x)). \end{array}$$

As both normal forms are equal, we have $\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathsf{s}(\mathcal{O}), x)) \equiv_{\mathcal{E}} \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), x)$.

Exercise 7 (Completion): (16 points)

In this exercise, we consider the signature $\Sigma = \{\mathsf{s}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{b}, \mathsf{c}\}$. Please use the **advanced** completion algorithm from the lecture to generate a convergent TRS that is equivalent to the following set of equations:

$$\{\ \mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x)),\ \ \mathsf{f}(x,y,z) \equiv \mathsf{g}(x,\mathsf{s}(z)),\ \ \mathsf{g}(x,y) \equiv \mathsf{b},\ \ \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\ \}$$

As reduction order $\succ$, use the LPO with precedence $\mathsf{s} \sqsupset \mathsf{f} \sqsupset \mathsf{g} \sqsupset \mathsf{h} \sqsupset \mathsf{b} \sqsupset \mathsf{c}$. For each step of the advanced completion algorithm, also indicate which transformation rule you are applying. In this exercise you do not need to give a proof for $\ell \succ r$ if you generate a new rule $\ell \to r$ (but this statement should be true, of course).

Solution:

$$\begin{array}{c|l}
\{\mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x)), \mathsf{f}(x,y,z) \equiv \mathsf{g}(x,\mathsf{s}(z)), \mathsf{g}(x,y) \equiv \mathsf{b}, \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \emptyset & \\ \hline
 & 1 \\
\{\mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x)), \mathsf{f}(x,y,z) \equiv \mathsf{g}(x,\mathsf{s}(z)), \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \{\mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 2 \\
\{\mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x)), \mathsf{f}(x,y,z) \equiv \mathsf{b}, \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \{\mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 3 \\
\{\mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x)), \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \{\mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 4 \\
\{\mathsf{b} \equiv \mathsf{h}(x,\mathsf{s}(x)), \mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \{\mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 5 \\
\{\mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)\}, \{\mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}, \mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 6 \\
\emptyset, \{\mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y), \mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}, \mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 7 \\
\{\mathsf{c}(\mathsf{s}(\mathsf{b})) \equiv \mathsf{b}\}, \{\mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y), \mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}, \mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} & \\ \hline
 & 8 \\
\emptyset, \{\mathsf{c}(\mathsf{s}(\mathsf{b})) \to \mathsf{b}, \mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y), \mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}, \mathsf{f}(x,y,z) \to \mathsf{b}, \mathsf{g}(x,y) \to \mathsf{b}\} &
\end{array}$$

Step 1: **Orient** term equation $\mathsf{g}(x,y) \equiv \mathsf{b}$, resulting in $\mathsf{g}(x,y) \to \mathsf{b}$.
Step 2: **Reduce** the right-hand side of term equation $\mathsf{f}(x,y,z) \equiv \mathsf{g}(x,\mathsf{s}(z))$, resulting in $\mathsf{f}(x,y,z) \equiv \mathsf{b}$.
Step 3: **Orient** term equation $\mathsf{f}(x,y,z) \equiv \mathsf{b}$, resulting in the rule $\mathsf{f}(x,y,z) \to \mathsf{b}$.
Step 4: **Reduce** the left-hand side of term equation $\mathsf{f}(x,x,z) \equiv \mathsf{h}(x,\mathsf{s}(x))$, resulting in $\mathsf{b} \equiv \mathsf{h}(x,\mathsf{s}(x))$.
Step 5: **Orient** term equation $\mathsf{b} \equiv \mathsf{h}(x,\mathsf{s}(x))$, resulting in $\mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}$.
Step 6: **Orient** term equation $\mathsf{c}(y) \equiv \mathsf{h}(\mathsf{b},y)$, resulting in $\mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y)$.
Step 7: **Generate** term equation $\mathsf{c}(\mathsf{s}(\mathsf{b})) \equiv \mathsf{b}$ for the (only) critical pair of $\mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y)$ and $\mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b}$.
Step 8: **Orient** term equation $\mathsf{c}(\mathsf{s}(\mathsf{b})) \equiv \mathsf{b}$, resulting in $\mathsf{c}(\mathsf{s}(\mathsf{b})) \to \mathsf{b}$.

After Step 8, all critical pairs of the persistent rules have been considered, and there are no remaining equations. Thus,

$$\{\ \mathsf{c}(\mathsf{s}(\mathsf{b})) \to \mathsf{b},\ \ \mathsf{h}(\mathsf{b},y) \to \mathsf{c}(y),\ \ \mathsf{h}(x,\mathsf{s}(x)) \to \mathsf{b},\ \ \mathsf{f}(x,y,z) \to \mathsf{b},\ \ \mathsf{g}(x,y) \to \mathsf{b}\ \}$$

is a convergent TRS which is equivalent to the term equation set from which we started.