Term Rewriting Systems SS 11
Exercise Sheet 1 (due April 20th, 2011)

Notes:

• Please solve these exercises in groups of two!

• Please register at https://aprove.informatik.rwth-aachen.de/tes11/ (https, not http!).

• The solutions must be handed in directly before (very latest: at the beginning of) the exercise course on Wednesday, April 20th, 2011, in lecture hall AH 2. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl’s office (until the exercise course starts).

• Please write the names and immatriculation numbers of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Syntax and Semantics): (2 + 4 = 6 points)

a) Give a set of equalities that describes the function of the arithmetic operators – as minus and / as div for subtraction and division on natural numbers. The exact semantics should be as follows:

\[
\begin{align*}
\text{div}(x, y) &= \left\lceil \frac{x}{y} \right\rceil & \text{if } y > 0 \\
\text{minus}(x, y) &= \begin{cases} 
    x - y & \text{if } x > y \\
    0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Thus, \(\text{minus}(15, 10) = 5\), \(\text{minus}(5, 10) = 0\) and \(\text{div}(15, 10) = 2\). You should not define \(\text{div}\) for a divisor of 0.

Use the representation of natural numbers presented in the lecture, where 0 is represented by \(O\) and \(n\) is represented by applying the successor symbol \(s\) \(n\) times (i.e., by \(s^n(O)\)). Define the signature of your equalities explicitly by giving definitions for \(\Sigma_i\) and \(\Sigma\).

b) Let \(\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2\) with \(\Sigma_0 = \{O\}, \Sigma_1 = \{s\}\) and \(\Sigma_2 = \{\text{plus}\}\). Consider \(E = \{\text{plus}(O, y) \equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y))\}\), the set of equations describing + on our representation of natural numbers.

Prove that \(E \not\models \text{plus}(x, y) \equiv \text{plus}(y, x)\).

Hints:

• You can use a model \(A = (\mathcal{A}, \alpha)\) where \(\mathcal{A}\) does not only consist of \(\mathbb{N}\), but also contains additional elements \(\Box\) and \(\Diamond\). Then define \(\alpha_{\text{plus}}(n, m)\) such that it models addition for \(n, m \in \mathbb{N}\), but behaves differently if \(n\) or \(m\) are from \(\{\Box, \Diamond\}\).

Exercise 2 (Matching): (2 + 2 = 4 points)

a) Consider the following pairs of terms \(s\) and \(t\) over the signature \(\Sigma = \Sigma_0 \cup \Sigma_2\) with \(\Sigma_0 = \{a\}\) and \(\Sigma_2 = \{f\}\). Moreover, we have \(\{x, y, z\} \subset \mathcal{V}\) for the set of variables \(\mathcal{V}\). If \(s\) matches \(t\), then give a suitable matcher \(\sigma\). Otherwise give a brief (at most two sentences) explanation why there is no matcher.

1. \(s = f(y, z), t = f(a, x)\)
2. \(s = f(x, a), t = f(a, x)\)
3. \(s = f(y, y), t = f(a, x)\)
4. \(s = f(x, y), t = f(f(x, y), a)\)

b) Let \(\sim\) be the matching relation, i.e., for two terms \(s\) and \(t\) we have \(s \sim t\) iff \(s\) matches \(t\).

Prove or disprove the following propositions:
1. For all terms \( s \) and \( t \) we have \( s \sim t \) iff \( t \sim s \) (i.e., the matching relation is symmetric).
2. For all terms \( s \) and \( t \) we have \( s \sim t \) and \( t \sim s \) iff \( s = t \).

Exercise 3 (Stability): \((1 + 1 + 1 + 1 = 4 \text{ points})\)

Consider the following relations \( \sim_1, \ldots, \sim_4 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V}) \). Prove or disprove for each of these relations that they are stable.

a) \( s \sim_1 t \) iff the numbers of different variables in \( s \) and \( t \) are equal, i.e., \( |V(s)| = |V(t)| \).

b) \( s \sim_2 t \) iff \( s \) is a subterm of \( t \).

c) \( s \sim_3 t \) iff \( s \) matches \( t \).

d) \( s \sim_4 t \) iff \( V(s) \subseteq V(t) \)

Hints:
- You can use the lemma proven in Exercise 4.

Exercise 4 (Induction): \((4 \text{ points})\)

Let \( t \in \mathcal{T}(\Sigma, \mathcal{V}) \), \( \pi \in \text{Occ}(t) \) and \( \sigma \in \text{SUB}(\Sigma, \mathcal{V}) \). Show by induction over \( \pi \) that \((t|_\pi)\sigma = (t\sigma)|_\pi \) holds.

Hints:
- In the induction base, prove the proposition for \( \pi = \epsilon \).
- In the induction step, consider the case \( \pi = i\pi' \), where as induction hypothesis, you can assume that \((q|_{i\pi'})\mu = (q\mu)|_{i\pi'} \) for all \( q \in \mathcal{T}(\Sigma, \mathcal{V}) \) and all \( \mu \in \text{SUB}(\Sigma, \mathcal{V}) \).