Exercise Sheet 1 (due April 20th, 2011)

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Notes:

- Please solve these exercises in **groups of two**!
- Please register at `https://aprove.informatik.rwth-aachen.de/tes11/` (http**s**, not http!).
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, April 20th, 2011, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Syntax and Semantics): (2 + 4 = 6 points)

a) Give a set of equalities that describes the function of the arithmetic operators $-$ as minus and $/$ as div for subtraction and division on natural numbers. The exact semantics should be as follows:

$$\mathsf{div}(x, y) = \lceil x/y \rceil \qquad \text{if } y > 0$$

$$\mathsf{minus}(x, y) = \begin{cases} x - y & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\mathsf{minus}(15, 10) = 5$, $\mathsf{minus}(5, 10) = 0$ and $\mathsf{div}(15, 10) = 2$. You should not define div for a divisor of 0.

Use the representation of natural numbers presented in the lecture, where 0 is represented by $\mathcal{O} \in \Sigma_0$ and n is represented by applying the successor symbol $\mathsf{s} \in \Sigma_1$ n times (i.e., by $\mathsf{s}^n(\mathcal{O})$). Define the signature of your equalities explicitly by giving definitions for Σ_i and Σ.

b) Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$ and $\Sigma_2 = \{\mathsf{plus}\}$. Consider $\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$, the set of equations describing $+$ on our representation of natural numbers.

Prove that $\mathcal{E} \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$.

Hints:

- You can use a model $A = (\mathcal{A}, \alpha)$ where $\mathcal{A}$ does not only consist of $\mathbb{N}$, but also contains additional elements $\Box$ and $\diamond$. Then define $\alpha_{\mathsf{plus}}(n, m)$ such that it models addition for $n, m \in \mathbb{N}$, but behaves differently if n or m are from $\{\Box, \diamond\}$.

Exercise 2 (Matching): (2 + 2 = 4 points)

a) Consider the following pairs of terms s and t over the signature $\Sigma = \Sigma_0 \cup \Sigma_2$ with $\Sigma_0 = \{\mathsf{a}\}$ and $\Sigma_2 = \{\mathsf{f}\}$. Moreover, we have $\{x, y, z\} \subset \mathcal{V}$ for the set of variables $\mathcal{V}$. If s matches t, then give a suitable matcher σ. Otherwise give a brief (at most two sentences) explanation why there is no matcher.

1. $s = \mathsf{f}(y, z)$, $t = \mathsf{f}(\mathsf{a}, x)$
2. $s = \mathsf{f}(x, \mathsf{a})$, $t = \mathsf{f}(\mathsf{a}, x)$
3. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, x)$
4. $s = \mathsf{f}(x, y)$, $t = \mathsf{f}(\mathsf{f}(x, y), \mathsf{a})$

b) Let $\sim$ be the matching relation, i.e., for two terms s and t we have $s \sim t$ iff s matches t.

Prove or disprove the following propositions:

1. For all terms s and t we have $s \sim t$ iff $t \sim s$ (i.e., the matching relation is symmetric).
2. For all terms s and t we have $s \sim t$ and $t \sim s$ iff $s = t$.

Exercise 3 (Stability): (1 + 1 + 1 + 1 = 4 points)

Consider the following relations $\sim_1, \ldots, \sim_4 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are stable.

a) $s \sim_1 t$ iff the numbers of different variables in s and t are equal, i.e., $|\mathcal{V}(s)| = |\mathcal{V}(t)|$.

b) $s \sim_2 t$ iff s is a subterm of t

c) $s \sim_3 t$ iff s matches t

d) $s \sim_4 t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$

Hints:

- You can use the lemma proven in Exercise 4.

Exercise 4 (Induction): (4 points)

Let $t \in \mathcal{T}(\Sigma, \mathcal{V})$, $\pi \in Occ(t)$ and $\sigma \in SUB(\Sigma, \mathcal{V})$. Show by induction over π that $(t|_\pi)\sigma = (t\sigma)|_\pi$ holds.

Hints:

- In the induction base, prove the proposition for $\pi = \epsilon$.
- In the induction step, consider the case $\pi = i\,\pi'$, where as induction hypothesis, you can assume that $(q|_{\pi'})\mu = (q\mu)|_{\pi'}$ for all $q \in \mathcal{T}(\Sigma, \mathcal{V})$ and all $\mu \in SUB(\Sigma, \mathcal{V})$.