Notes:

- Please solve these exercises in groups of two!
- The solutions must be handed in directly before (very latest: at the beginning of) the exercise course on Wednesday, May 4th, 2011, in lecture hall AH 2. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl’s office (until the exercise course starts).
- Please write the names and immatriculation numbers of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Monotonicity): (1 + 1 + 1 + 1 = 4 points)

Consider the following relations \( \sim_1, \ldots, \sim_4 \subseteq T(\Sigma, \mathcal{V}) \times T(\Sigma, \mathcal{V}) \). Prove or disprove for each of these relations that they are monotonic.

a) \( s \sim_1 t \) iff the numbers of different variables in \( s \) and \( t \) are equal, i.e., \( |\mathcal{V}(s)| = |\mathcal{V}(t)| \).

b) \( s \sim_2 t \) iff \( s \) is a subterm of \( t \).

c) \( s \sim_3 t \) iff \( s \) matches \( t \).

d) \( s \sim_4 t \) iff \( \mathcal{V}(s) \subseteq \mathcal{V}(t) \).

Exercise 2 (Equivalence relations): ((1 + 1 + 1 + 1) + (1 + 1) = 6 points)

a) Consider the following relations \( \sim_1, \ldots, \sim_4 \). Prove or disprove for each of these relations that they are equivalence relations.

i) \( \sim_1 \subseteq T(\Sigma, \mathcal{V}) \times T(\Sigma, \mathcal{V}) \) with \( s \sim_1 t \iff |\mathcal{V}(s)| = |\mathcal{V}(t)| \).

ii) \( \sim_2 \subseteq T(\Sigma, \mathcal{V}) \times T(\Sigma, \mathcal{V}) \) with \( s \sim_2 t \iff \exists \sigma \in \text{SUB}(\Sigma, \mathcal{V}) : s\sigma = t\sigma \) (i.e., \( s \) and \( t \) unify).

iii) Let \( m \in \mathbb{N} \) be some fixed natural number. Then \( \sim_3 \subseteq \mathbb{Z} \times \mathbb{Z} \) with \( x \sim_3 y \iff (x - y) \mod m = 0 \).

For example, we have \( 61 \sim_3^{10} 23 \), as \( (61 - 23) \mod 10 = 38 \mod 19 = 0 \).

Hints:
- To save work, you may write \( x - y \equiv_m 0 \) instead of \( (x - y) \mod m = 0 \).

iv) Let \( \mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\} \) and \( (p, q), (u, v) \in \mathbb{Z} \times \mathbb{Z}_{\neq 0} \). Then \( \sim_4 \subseteq (\mathbb{Z} \times \mathbb{Z}_{\neq 0}) \times (\mathbb{Z} \times \mathbb{Z}_{\neq 0}) \) with \( (p, q) \sim_4 (u, v) \iff p \cdot v = u \cdot q \).

For example, we have \( (3, 6) \sim_4 (2, 4) \), as \( 3 \cdot 4 = 12 = 2 \cdot 6 \).

b) For each of the relations \( \sim_1 \) and \( \sim_2 \), give the smallest equivalence relation (i.e., the transitive-reflexive-symmetric closure) that includes them.

i) Let \( M = \{0, 1, 2, 3, 4\} \) and \( 0 \sim_1 2, 3 \sim_1 1, 4 \sim_1 2 \) and \( 2 \sim_1 0 \).

ii) Let \( \mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\} \). Then \( \sim_2 \subseteq \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\neq 0} \) with \( x \sim_2 y \iff x + 1 = y \).
Exercise 3 (Equivalence classes):

(2 + 4 = 6 points)

a) Let \( s \sim t \) hold for two terms \( s \) and \( t \) iff \( V(s) = V(t) \) and the number of function symbols in \( s \) is the same as the number of function symbols in \( t \).

Please show that \( \sim \) is an equivalence relation and that all equivalence classes w.r.t. \( \sim \) are finite.

b) Please show that the word problem is decidable for the following set of equations \( E \) over \( \Sigma = \Sigma_2 \cup \Sigma_0 \) with \( \Sigma_2 = \{ :, \cup \} \) and \( \Sigma_0 = \{ a \} \).

\[
\begin{align*}
(x : y) \cup z &\equiv x : (y \cup z) \\
x \cup (y \cup z) &\equiv (x \cup y) \cup z \\
x \cup y &\equiv y \cup x \\
x : (y : z) &\equiv y : (x : z) \\
x : (y \cup z) &\equiv y \cup (x : z)
\end{align*}
\]

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

Exercise 4 (Syntactic Proofs):

(1 + 4 = 5 points)

Consider the following set of equations\(^1\) \( E \):

\[
\begin{align*}
f(x, f(y, z)) &\equiv f(f(x, y), z) \\
f(x, e) &\equiv x \\
f(x, i(x)) &\equiv e \\
f(i(x), x) &\equiv e
\end{align*}
\]

(1) \begin{align*}
(2) \\
(3) \\
(3)'
\end{align*}

a) Prove \( f(e, x) \equiv x \) using \( \leftrightarrow^*_E \). Mark in each step which part of your term you are replacing and which equation you used for it.

b) Prove \( f(i(v), i(u)) \equiv i(f(u, v)) \) using \( \leftrightarrow^*_E \). Mark in each step which part of your term you are replacing and which equation you used for it.

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\(^1\)They correspond to the group axioms from the lecture and an additional equation for the operation of inverse elements from the right hand side.