Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (3 + 2 + 2 = 7 points)
Consider the following set of equalities $\mathcal{E}$ and the term rewrite system $\mathcal{R}$.

\[ \mathcal{E} = \{ f(f(g(x))) \equiv x, x \equiv g(x), f(g(x)) \equiv g(g(f(x))) \} \]
\[ \mathcal{R} = \{ f(f(x)) \rightarrow x, g(x) \rightarrow x \} \]

a) Please show that $\mathcal{R}$ is equivalent to $\mathcal{E}$.

b) Please show $f(f(f(f(f(f(f(f(f(f(x)))))))))) \equiv_E f(g(f(f(x))))$ only using the relation $\leftrightarrow_E$ and Birkhoff’s Theorem (in particular, you must not use $\mathcal{R}$ in this subexercise).

c) Please show $f(f(f(f(f(f(f(f(f(x)))))))))) \equiv_E f(g(f(f(x))))$ using the algorithm WORTPROBLEM.

Hints:
- $\mathcal{R}$ is convergent.

Exercise 2 (Noetherian Induction): (2 + 4 = 6 points)
Consider the following term rewrite system $\mathcal{R}$, which represents the well-known Ackermann function:

\[ \text{ack}(O, m) \rightarrow s(m) \quad (1) \]
\[ \text{ack}(s(n), O) \rightarrow \text{ack}(n, s(O)) \quad (2) \]
\[ \text{ack}(s(n), s(m)) \rightarrow \text{ack}(n, \text{ack}(s(n), m)) \quad (3) \]

a) Choose a relation $\triangleright \subseteq \{ (s^n(O), s^k(O)) \mid n_1, k_1 \in \mathbb{N} \} \times \{ (s^n(O), s^k(O)) \mid n_2, k_2 \in \mathbb{N} \}$ and prove that your $\triangleright$ is well-founded ("fundiert").

b) Prove that any normal form of $\text{ack}(s^n(O), s^m(O))$ has the form $s^k(O)$ by noetherian induction using the relation $\triangleright$ from part a).
Exercise 3 (The Algorithm RIGHT-GROUND TERMINATION): \( (3 + 2 = 5 \text{ points}) \)

Prove or disprove termination of the following term rewrite systems over the signature \( \Sigma = \{ f, a, b \} \) using the algorithm RIGHT-GROUND TERMINATION from the lecture:

a) 

\[
\begin{align*}
f(f(x, y), z) & \rightarrow f(a, f(a, b)) \\
f(a, f(x, x)) & \rightarrow f(a, f(b, a)) \\
f(a, x) & \rightarrow a \\
f(x, b) & \rightarrow f(a, a) \\
f(b, a) & \rightarrow b
\end{align*}
\]

b) 

\[
\begin{align*}
f(a, f(a, x)) & \rightarrow f(a, a) \\
f(x, f(a, f(x, a))) & \rightarrow f(a, f(a, f(a, b))))
\end{align*}
\]